Applications of Calculus – Answer Sheet

Q1 Avg $= \dfrac{15-(-3)}{2} = 9$; $f'(2) = 3(2)^2 - 4 = 8$.

Q2 $g'(x) = \dfrac{1}{2\sqrt{x}} + 2$; $g'(1) = \frac{1}{2} + 2 = \frac{5}{2}$.

Q3 $T(0) = 44$, $T(5) = 28 + 16e^{-1.5} = 31.570$; avg rate $= \dfrac{31.570 - 44}{5} = -2.486$ °C/min.
$T'(t) = -4.8e^{-0.3t} \Rightarrow T'(5) = -4.8e^{-1.5} = -1.071$ °C/min.

Q4 $y'(x) = 3x^2 - 12x + 9$; $y'(3) = 0$, point $(3,4)$. Tangent $y = 4$; normal $x = 3$.

Q5 $y' = \dfrac{(x-2)-(x+1)}{(x-2)^2} = -\dfrac{3}{(x-2)^2}$; at $x = 3$: gradient -3.
Normal gradient $= 1/3$; $y - 4 = \frac{1}{3}(x-3)$.

Q6 $y'(1) = 2(1) + \dfrac{1}{2} = \frac{5}{2}$; tangent $y - 3 = \frac{5}{2}(x-1)$.

Q7 $f'(x) = 3(x-1)(x-3) \Rightarrow x = 1, 3$. $f''(x) = 6x - 12$. $f''(1) = -6$ max at $(1,8)$; $f''(3) = 6$ min at $(3,4)$.

Q8 $y' = 4x(x^2-2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$. $y'' = 12x^2 - 8$.
$x = 0$: $y'' < 0 \Rightarrow$ local max $(0,0)$. $x = \pm\sqrt{2}$: $y'' > 0$ minima $(\pm\sqrt{2}, -4)$.
Inflections: $y'' = 0 \Rightarrow x = \pm\sqrt{2/3}$.

Q9 $y' = e^{-x}(1-x) = 0 \Rightarrow x = 1$ (on $x \geq 0$). Max value $y(1) = 1/e$.

Q10 $A(x) = 2x(12 - x^2) = 24x - 2x^3$, $A'(x) = 24 - 6x^2 = 0 \Rightarrow x = 2$. Width 4, height 8, $A_{\max} = 32$.

Q11 $h = \dfrac{500}{\pi r^2}$; $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{1000}{r}$.
$S'(r) = 4\pi r - \dfrac{1000}{r^2} = 0 \Rightarrow r^3 = \dfrac{250}{\pi} \Rightarrow r \approx 4.29$ cm.

Q12 Perimeter fencing $x + 2y = 200$; $A = xy = x(200-x)/2 = 100x - \frac{1}{2}x^2$.
$A'(x) = 100 - x = 0 \Rightarrow x = 100$, $y = 50$ (max).

Q13 Arc $= \theta r = 2\pi R \Rightarrow R = \dfrac{\theta}{2\pi} r$. Volume $V = \frac{1}{3}\pi R^2 \sqrt{r^2 - R^2}$.
Max when $R = r\sqrt{2/3} \Rightarrow \theta = 2\pi\sqrt{2/3}$ ($\approx 294.0°$).

Q14 $R = pq = 80q - 0.4q^2$; $P = R - C = -0.4q^2 + 70q - 120$.
$P'(q) = -0.8q + 70 = 0 \Rightarrow q = 87.5$; $P_{\max} = -0.4(87.5)^2 + 70(87.5) - 120 = \$2942.50$.

Q15 $h'(t) = -k\sqrt{h}$. (a) $dh/dt = -0.3 \cdot 2 = -0.6$ m/min at $h = 4$.
(b) $dh/\sqrt{h} = -k\,dt \Rightarrow 2\sqrt{h} = -kt + C$. $h(0) = 9 \Rightarrow C = 6$.
$\sqrt{h} = 3 - \dfrac{k}{2}t \Rightarrow h(t) = \big(3 - 0.15t\big)^2$ (until empty at $t = 20$).

Q16 (a) $dA/dt = 2\pi r\,dr/dt = 2\pi(15)(0.6) = 18\pi$ m^2/min.
(b) $s = 2\pi r \Rightarrow ds/dt = 2\pi(0.6) = 1.2\pi$ m/min.

Q17 $h = \frac{6}{5}r \Rightarrow r = \frac{5}{6}h$. $V = \frac{1}{3}\pi r^2 h = \frac{25}{108}\pi h^3$.
$dV/dt = \frac{25}{36}\pi h^2\,dh/dt$; at $h = 30$, $dV/dt = \frac{25}{36}\pi(900)(2.4) = 1500\pi$ cm^3/min.

Q18 $x^2 + y^2 = 4.8^2$, $dx/dt = 0.35$.
(a) $dy/dt = -(x/y)\,dx/dt$; at $x = 2$, $y = \sqrt{23.04 - 4} = 4.362 \Rightarrow dy/dt = -0.160$ m/s.
(b) $x = 4.8\cos\theta \Rightarrow dx/dt = -4.8\sin\theta\,d\theta/dt$.
$\sin\theta = y/4.8 = 0.9088 \Rightarrow d\theta/dt = -0.35/(4.8 \cdot 0.9088) = -0.0802$ rad/s.

Q19 Similar triangles: $\dfrac{6}{x+s} = \dfrac{1.8}{s} \Rightarrow 4.2s = 1.8x \Rightarrow s = \frac{3}{7}x$.
$ds/dt = \frac{3}{7}(1.5) = 0.6429$ m/s; tip speed $= dx/dt + ds/dt = 2.1429$ m/s.

Q20 Displacement $\displaystyle\int_0^6 v\,dt = 18$ m. Speed changes at $t = 2, 3$.
Distance $= \frac{14}{3} + \frac{13}{6} + \frac{27}{2} = \frac{61}{3} \approx 20.33$ m.

Q21 $v = 3(t-2)(t-4)$. Right: $t < 2$ and $t > 4$; left: $2 < t < 4$.
$s(0) = 0,\ s(2) = 20,\ s(4) = 16,\ s(8) = 128 \Rightarrow$ maximum displacement $= 128$ m at $t = 8$.

Q22 $a = 6t - 12$; $v = 3t^2 - 12t + 8$; $s = t^3 - 6t^2 + 8t + 3$.
v minimal when $a = 0 \Rightarrow t = 2$.

Q23 Roots $0, 4$. Area $\displaystyle\int_0^4 (4x - x^2)\,dx = \frac{32}{3}$.

Q24 Intersections $0, 4$; area $= \displaystyle\int_0^4 (4x - x^2)\,dx = \frac{32}{3}$.

Q25 Intersections: $x = -2, 3$. Area $= \displaystyle\int_{-2}^{3} \big((x+2) - (x^2 - 4)\big)dx = \frac{125}{6}$.

Q26 $f_{\text{avg}} = \dfrac{1}{6}\int_0^6 (6x - x^2)\,dx = 6$; $\quad f \geq 8$ on $[2, 4] \Rightarrow$ proportion $= 2/6 = 1/3$.

Q27 $\dfrac{1}{\pi}\int_0^\pi e^{-x}\sin x\,dx = \dfrac{1 + e^{-\pi}}{2\pi}$.

Q28 Similar triangles: width w scales linearly with depth h; $w(h) = 2h$.
Cross–sectional area $= \frac{1}{2}wh = h^2$. $V(h) = 5h^2$ m^3; $dV/dh = 10h$, so at $h = 0.5$, $dV/dh = 5$ m^3/m.

Q29 $y' = \dfrac{1 - x^2}{(x^2 + 1)^2}$. Stat. pts $x = \pm 1$: $(-1, -\frac{1}{2})$ min, $(1, \frac{1}{2})$ max.
Normal slope $= -1$ when $y' = 1 \Rightarrow x = 0$, point $(0, 0)$.

Q30 $y' = e^{-x}(2x - x^2) = 0 \Rightarrow x = 0, 2$; maximum at $x = 2$ with $y = 4e^{-2}$.

Q31 $h = 108/x^2$; $S = x^2 + 4xh = x^2 + \dfrac{432}{x}$.
$S'(x) = 2x - \dfrac{432}{x^2} = 0 \Rightarrow x^3 = 216 \Rightarrow x = 6$ cm; $h = 3$ cm.

Q32 $C(q) = \int(0.03q^2 - 0.9q + 15)\,dq + 120 = 0.01q^3 - 0.45q^2 + 15q + 120.$
Marginal min: $C''(q) = 0.06q - 0.9 = 0 \Rightarrow q = 15.$

Q33 $\int_0^4 k(4-t)\,dt = 8k = 18 \Rightarrow k = 2.25.$
$s(t) = 2.25(4t - \frac{t^2}{2})$. $s = 12 \Rightarrow 4t - \frac{t^2}{2} = 16/3$
$\Rightarrow t^2 - 8t + \frac{32}{3} = 0 \Rightarrow t = 4 \pm \frac{4}{\sqrt{3}}$; valid $t = 4 - \frac{4}{\sqrt{3}} \approx 1.691$ s.

Challenge — Designing a rainwater tank (Detailed)

Required volume $V = 10$ m^3: $\pi r^2 h = 10 \Rightarrow h = \dfrac{10}{\pi r^2}$.

(a) Cost function

Side area $= 2\pi r h$ at \$120/m^2: $C_{\text{side}} = 120(2\pi r h) = 240\pi r h = 240\pi r \cdot \dfrac{10}{\pi r^2} = \dfrac{2400}{r}$.
Base+top area $= 2\pi r^2$ at \$180/m^2: $C_{\text{ends}} = 180(2\pi r^2) = 360\pi r^2$.
Total

$$C(r) = 360\pi r^2 + \frac{2400}{r}, \qquad r > 0.$$

(b) Optimal r and h

$$C'(r) = 720\pi r - \frac{2400}{r^2} = 0 \;\Rightarrow\; 720\pi r^3 = 2400 \;\Rightarrow\; r^3 = \frac{10}{3\pi}.$$

$$r = \left(\tfrac{10}{3\pi}\right)^{1/3} \approx 1.020 \text{ m}, \qquad h = \frac{10}{\pi r^2} \approx 3.061 \text{ m}.$$

Since $C''(r) = 720\pi + \dfrac{4800}{r^3} > 0$, this is a minimum.

(c) Minimum cost

$$C_{\min} = 360\pi r^2 + \frac{2400}{r} = 360\pi(1.0201)^2 + \frac{2400}{1.0201} \approx 1176.58 + 2352.24 \approx \boxed{\$3{,}529}.$$

(d) Side cost \$140/m^2

Now $C(r) = 360\pi r^2 + \dfrac{2800}{r}$.

$$C'(r) = 720\pi r - \frac{2800}{r^2} = 0 \;\Rightarrow\; r^3 = \frac{2800}{720\pi} = \frac{35}{9\pi} \approx 1.238,$$

$$\boxed{r \approx 1.074 \text{ m}}, \qquad \text{increase } \approx 1.074 - 1.020 = 0.054 \text{ m (about 5.3\%)}.$$