Sequences and Series – Question Set

Arithmetic progressions

Q1 In an AP, $u_1 = 7$, $d = 3$. Find u_{20} and S_{20}.

Q2 An AP has $u_5 = 18$ and $u_{13} = 50$. Find u_1 and d.

Q3 The nth term is $u_n = 4n - 7$. Find u_1 and d.
Find S_n in terms of n.

Q4 The sum of the first n terms is $S_n = 2n^2 - 5n$. Find a formula for u_n.
Hence find u_{25}.

Q5 An AP has positive terms. If $S_{20} = 710$ and $u_{20} = 64$, find u_1 and d.

Q6 In an AP, $S_n = 990$ and $S_{2n} = 2\,310$. Find u_1 and d.

Geometric progressions

Q7 A GP has $u_1 = 250$, $r = 0.8$. Find u_8 and S_8.

Q8 For a GP, $u_3 = 45$ and $u_6 = 360$. Find u_1 and r.

Q9 A GP has $u_1 = 12$, $r = \frac{2}{3}$. Find S_∞.
Find the least n with $S_n > 17$.

Q10 A GP with positive terms satisfies $S_\infty = 40$ and $u_3 = 9.6$. Find u_1 and r.

Q11 If $u_1 + u_2 + u_3 = 14$ and $u_1 u_2 u_3 = 27$ for a GP with $u_1 > 0$, find u_1, u_2, u_3.

Q12 Show that for $|r| < 1$,

$$\sum_{k=1}^{n} k r^{k-1} = \frac{1 - (n+1)r^n + n r^{n+1}}{(1-r)^2}.$$

Sigma notation and telescoping

Q13 Evaluate $\displaystyle\sum_{k=1}^{20}(3k^2 - 2k + 1)$.

Q14 Show that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$ and hence compute the sum for $n = 50$.

Q15 Evaluate $\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$ and simplify.

Q16 Find $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)}$.

Q17 Evaluate $\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right)$ and deduce a closed form.

Q18 Compute $\sum_{k=1}^{10} k(0.8)^{k-1}$.

Recurrences

Q19 Solve $u_{n+1} = 1.5\,u_n + 4$ with $u_0 = 2$.

Q20 Solve $u_{n+1} = 0.7\,u_n + 12$ with $u_1 = 5$. Find $\lim_{n\to\infty} u_n$.

Q21 A sequence satisfies $u_{n+1} - u_n = 6$ with $u_3 = 17$. Find u_n.

Q22 Let $u_{n+1} = au_n + b$ with $a \neq 1$. Show that $u_n = (u_0 - u^*)a^n + u^*$, where $u^* = \frac{b}{1-a}$.

Q23 The sequence v_n is defined by $v_1 = 3$, $v_{n+1} = 2v_n + 1$. Find v_n.

Limits and convergence

Q24 For $u_{n+1} = 0.6u_n + 9$ with $u_0 = 40$, find $\lim u_n$ and the least n such that $u_n < 25$.

Q25 Suppose $x_{n+1} = \frac{x_n + 6}{2}$ with $x_1 = 0$. Show x_n is increasing and find $\lim x_n$.

Q26 Let $a_n = 3 + \frac{5}{2^n} - \frac{1}{3^n}$. Determine $\lim a_n$ and whether the sequence is monotone.

Mixed practice

Q27 In an AP, $u_p = 17$ and $u_q = 41$ with $p < q$. Show that $d = \frac{24}{q-p}$ and find u_1.

Q28 For a GP of positive terms, $u_m u_n = 400$ and $u_{m+2}u_{n-2} = 400$ with $n > m+2$. Show that $r = 1$ or $r^2 = 1$, and deduce the only possible ratio.

Q29 If S_n is the sum to n terms of an AP with $S_5 = 40$ and $S_8 = 88$, find u_1 and d.

Q30 Show that $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$ by considering $(k+1)^4 - k^4$.

Q31 Find the value of n for which the sum $5 + 9 + 13 + \cdots$ first exceeds $1\,000$.

Q32 Evaluate $\sum_{k=1}^{n} \frac{k}{(k+1)!}$ and simplify to a form not involving a summation.

Q33 Let $S = \sum_{k=1}^{\infty} \frac{2}{3^k}$ and $T = \sum_{k=1}^{\infty} \frac{k}{3^k}$. Find S and T.

Q34 Prove by induction that $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

Challenge Question

Q35 Arithmetic–geometric series and optimisation
A tank is filled by pulses of water. The kth pulse adds a_k litres where (a_k) forms a GP with $a_1 = 40$ and ratio r $(0 < r < 1)$. Between pulses, a fixed 5 litres leaks out. Exactly n pulses are delivered, with enough time between pulses for the leakage to occur fully each time.

(a) Show that the final volume after n pulses is

$$V_n = 40\frac{1 - r^n}{1 - r} - 5n.$$

(b) For fixed n, find the value of $r \in (0, 1)$ that maximises V_n. (Hint: differentiate with respect to r.)

(c) For $n = 8$, find the optimal r (to 3 d.p.) and the corresponding maximum final volume (nearest litre).

(d) Suppose instead the leakage is proportional to the current volume, leaking a fraction λ $(0 < \lambda < 1)$ between pulses. The recurrence becomes

$$V_{k+1} = (1 - \lambda)\, V_k + a_{k+1}, \qquad a_{k+1} = 40r^k.$$

Show that

$$V_n = (1 - \lambda)^n V_0 + \sum_{j=0}^{n-1} 40r^j (1 - \lambda)^{n-1-j},$$

and hence give V_n in closed form as a finite geometric-convolution sum.