Further Graph Transformations and Modelling – Worked Examples

Key Facts / Formulas

- Master transformation: $y = A\,f\big(B(x-h)\big) + k$.
  Horizontal shift h right; vertical shift k up; vertical dilation factor $|A|$ (reflection in x-axis if $A<0$); horizontal dilation factor $1/|B|$ (reflection in y-axis if $B<0$).
- Domains transform via $x \mapsto \dfrac{x-h}{B}$; ranges multiply by A and shift by k.
- Vertical asymptotes of f at $x=a$ map to $B(x-h)=a \;\Rightarrow\; x = h + \dfrac{a}{B}$. Horizontal asymptote $y=L$ maps to $y = AL+k$.
- Reciprocal forms: $y = \dfrac{p}{x-q} + r$ has asymptotes $x=q$ and $y=r$; $y = \dfrac{p}{(x-q)^2} + r$ is always above/below r depending on sign(p).
- Absolute value: $|u| = \begin{cases} u, & u \ge 0 \\ -u, & u < 0 \end{cases}$; reflect the negative part of u in the x-axis.
- Log-linearisation:
  Exponential $y = ab^x \;\Rightarrow\; \ln y = \ln a + x\ln b$ (linear in x).
  Power $y = kx^n \;\Rightarrow\; \ln y = \ln k + n\ln x$ (linear in $\ln x$).
  Logarithmic $y = a + b\ln x$ is already linear in $\ln x$.

Example 1 Mapping asymptotes and intercepts under A, B, h, k

Let $f(x) = \dfrac{1}{x}$, with asymptotes $x=0$, $y=0$. Consider

$$y = g(x) = -3\,f\big(2(x-4)\big) + 5 = -3\cdot\frac{1}{2(x-4)} + 5 = -\frac{3}{2(x-4)} + 5.$$

Asymptotes. Vertical: $2(x-4)=0 \Rightarrow x=4$. Horizontal: $y = A\cdot 0 + k = 5$.

Intercepts. x-intercept: solve $0 = -\dfrac{3}{2(x-4)} + 5 \Rightarrow \dfrac{3}{2(x-4)} = 5 \Rightarrow 2(x-4) = \dfrac{3}{5} \Rightarrow x = 4 + \dfrac{3}{10} = 4.3$.

y-intercept: $x=0$ gives $y = -\dfrac{3}{2(-4)} + 5 = \dfrac{3}{8} + 5 = \dfrac{43}{8} = 5.375$.

$$\boxed{\text{VA } x=4,\ \text{HA } y=5,\ x\text{-int } (4.3, 0),\ y\text{-int } (0, 5.375)}$$

Example 2 Domain and range under a transformation

Let $f(x) = \sqrt{x}$ with domain $[0,\infty)$, range $[0,\infty)$. Define

$$y = g(x) = 2\,f(1-3x) - 4 = 2\sqrt{1-3x} - 4.$$

Set $1 - 3x \geq 0 \Rightarrow x \leq \dfrac{1}{3}$. Thus $\text{dom}(g) = (-\infty, \frac{1}{3}]$.
For the range, set $u = 1 - 3x \in [0, \infty)$: $\sqrt{u} \in [0, \infty)$, so $2\sqrt{u} \in [0, \infty)$ and $2\sqrt{u} - 4 \in [-4, \infty)$. Therefore

$$\boxed{\text{dom} = (-\infty, \tfrac{1}{3}],\ \text{range} = [-4, \infty).}$$

Example 3 Absolute value as a piecewise transform

Sketch $y = |2x - 5| - 3$. Vertex occurs when $2x - 5 = 0 \Rightarrow x = 2.5$, then $y = -3$.
Piecewise:

$$y = \begin{cases} (2x-5) - 3 = 2x - 8, & x \geq 2.5, \\ -(2x-5) - 3 = -2x + 2, & x < 2.5. \end{cases}$$

Intercepts: y-intercept $x = 0 \Rightarrow y = |-5| - 3 = 2$. x-intercepts solve $|2x - 5| = 3$: $2x - 5 = 3 \Rightarrow x = 4$ or $2x - 5 = -3 \Rightarrow x = 1$.

$$\boxed{\text{Vertex } (2.5, -3),\ x\text{-ints } (1, 0), (4, 0),\ y\text{-int } (0, 2)}$$

Example 4 Fitting a rectangular hyperbola $y = \dfrac{p}{x - q} + r$

A graph has vertical asymptote $x = 3$ and horizontal asymptote $y = -2$, and passes through $(1, 1)$. Find p, q, r.
Form $y = \dfrac{p}{x - 3} - 2$. Sub $(1, 1)$:

$$1 = \frac{p}{1 - 3} - 2 = \frac{p}{-2} - 2 \Rightarrow \frac{p}{-2} = 3 \Rightarrow p = -6.$$

Thus

$$\boxed{y = -\frac{6}{x - 3} - 2.}$$

Example 5 Determining A, B, h, k from a point and asymptotes

A curve has the form $y = A\,\dfrac{1}{B(x - h)} + k$, with asymptotes $x = 2$ and $y = 1$. It passes through $(4, \frac{3}{2})$. Find A and B.
From asymptotes: $h = 2$, $k = 1$. Then

$$\frac{3}{2} = A\,\frac{1}{B(4 - 2)} + 1 = \frac{A}{2B} + 1 \Rightarrow \frac{A}{2B} = \frac{1}{2} \Rightarrow A = B.$$

Choose $A = B$ (nonzero). Any common nonzero value works; simplest $A = B = 1$. Hence

$$\boxed{y = \frac{1}{x - 2} + 1 \quad \text{(one convenient model)}}$$

(Any $A = B \neq 0$ gives the same point and asymptotes; data would be needed to fix scale.)

Example 6 Exponential model through two points

Suppose $y = ab^x$ fits $(0,4)$ and $(3,27)$. Find a, b and predict y when $x = 5$.

$(0,4)$ gives $a = 4$. Then $27 = 4b^3 \Rightarrow b^3 = \dfrac{27}{4} \Rightarrow b = \left(\dfrac{27}{4}\right)^{1/3} = \dfrac{3}{\sqrt[3]{4}}$. Prediction:

$$y(5) = 4b^5 = 4\left(\frac{27}{4}\right)^{5/3} = 4 \cdot \frac{3^5}{4^{5/3}} = \frac{972}{4^{2/3}} \approx 972\,(0.62996) \approx 612.$$

$$\boxed{a = 4,\ b = \left(\tfrac{27}{4}\right)^{1/3},\ y(5) \approx 6.12 \times 10^2}$$

Example 7 Power model via log–log linearisation

Resistance force F on a sphere varies with speed v as $F = kv^n$. Measurements: $(v, F) = (2, 18), (5, 140)$. Find k, n.

$\ln F = \ln k + n \ln v$. Slope using the two points:

$$n = \frac{\ln 140 - \ln 18}{\ln 5 - \ln 2} = \frac{\ln(140/18)}{\ln(5/2)} = \frac{\ln(7.777\ldots)}{\ln 2.5} \approx \frac{2.051}{0.9163} = 2.239.$$

Then $k = \dfrac{F}{v^n} = \dfrac{18}{2^{2.239}} = 18/4.724 \approx 3.81$. So

$$\boxed{F \approx 3.81\, v^{2.24}}$$

Example 8 Logarithmic model $y = a + b \ln x$ from two data points

Data: $(x, y) = (1, 4.2)$ and $(e, 6.0)$. Fit $y = a + b \ln x$.

At $x = 1$: $4.2 = a + b \cdot 0 \Rightarrow a = 4.2$. At $x = e$: $6.0 = 4.2 + b \cdot 1 \Rightarrow b = 1.8$.

$$\boxed{y = 4.2 + 1.8 \ln x}$$

Example 9 Transforming a restricted domain

Let $f(x) = \sqrt{4 - x}$ with domain $(-\infty, 4]$, range $[0, \infty)$. Consider $y = g(x) = -\dfrac{1}{2} f\left(\dfrac{x+1}{2}\right) + 3$.

Domain condition: $\dfrac{x+1}{2} \le 4 \Rightarrow x + 1 \le 8 \Rightarrow x \le 7$. Thus $\mathrm{dom}(g) = (-\infty, 7]$.

Range: $f(u) \in [0, \infty)$ so $-\dfrac{1}{2} f(u) \in (-\infty, 0]$, then add 3: $(-\infty, 3]$. Hence

$$\boxed{\text{dom } (-\infty, 7],\ \text{range } (-\infty, 3].}$$

Key points: when $u = 0$ i.e. $(x+1)/2 = 0 \Rightarrow x = -1$, then $g(-1) = 3$; when $u = 4$ $(x = 7)$, $g(7) = 3 - \frac{1}{2}\sqrt{0} = 3$; decreasing to the left.

Example 10 Reciprocal quadratic model with given extremum level

A curve of the form $y = \dfrac{a}{(x-1)^2} + c$ has a minimum value $y = 5$ attained at $x \neq 1$,

and passes through $(3, 6)$. Find a, c.
Since $(x-1)^2 > 0$ for $x \neq 1$, the sign of a decides whether y is above or below c. A minimum at value 5 implies $a > 0$ and $c = 5$ (approaches 5 from above as $|x-1| \to \infty$).
Use the point $(3, 6)$:

$$6 = \frac{a}{(3-1)^2} + 5 = \frac{a}{4} + 5 \Rightarrow a = 4.$$

$$\boxed{y = \frac{4}{(x-1)^2} + 5}$$