Sequences and Series – Answer Sheet

Q1 $u_{20} = 7 + 19 \cdot 3 = 64$; $S_{20} = \frac{20}{2}(7 + 64) = 710$.

Q2 $u_5 = u_1 + 4d = 18$, $u_{13} = u_1 + 12d = 50 \Rightarrow 8d = 32 \Rightarrow d = 4$, $u_1 = 18 - 16 = 2$.

Q3 $u_1 = -3$, $d = 4$; $S_n = \dfrac{n}{2}\big(2(-3) + (n-1)4\big) = n(2n-5)$.

Q4 $u_n = S_n - S_{n-1} = 4n - 7$; hence $u_{25} = 4 \cdot 25 - 7 = 93$.

Q5 $u_{20} = u_1 + 19d = 64$, and $S_{20} = 10(u_1 + u_{20}) = 710 \Rightarrow u_1 = 7$, $d = \dfrac{64-7}{19} = \dfrac{57}{19} = 3$.

Q6 Let $A = 2u_1$. From $S_n = \dfrac{n}{2}(A + (n-1)d) = 990$ and $S_{2n} = n(A + (2n-1)d) = 2310$. Hence

$$A + (n-1)d = \frac{1980}{n}, \qquad A + (2n-1)d = \frac{2310}{n}.$$

Subtract: $nd = \dfrac{330}{n} \Rightarrow d = \dfrac{330}{n^2}$ and

$$A = \frac{1980}{n} - (n-1)d = \frac{330(5n+1)}{n^2}, \quad u_1 = \frac{A}{2} = \frac{165(5n+1)}{n^2}.$$

(There are infinitely many (u_1, d) pairs parameterised by n.)

Q7 $u_8 = 250(0.8)^7 \approx 52.43$; $S_8 = 250\frac{1-0.8^8}{0.2} \approx 1040.28$.

Q8 $r^3 = \dfrac{u_6}{u_3} = \dfrac{360}{45} = 8 \Rightarrow r = 2$; $u_1 = \dfrac{u_3}{r^2} = \dfrac{45}{4} = 11.25$.

Q9 $S_\infty = \dfrac{12}{1 - 2/3} = 36$. $S_n = 36\big(1 - (2/3)^n\big) > 17 \Rightarrow (2/3)^n < 19/36$; $n > \dfrac{\ln(19/36)}{\ln(2/3)} \approx 1.576 \Rightarrow$ $n = 2$.

Q10 $S_\infty = \dfrac{u_1}{1-r} = 40 \Rightarrow u_1 = 40(1-r)$; $u_3 = u_1 r^2 = 40(1-r)r^2 = 9.6 \Rightarrow r^2(1-r) = 0.24$. But $\max_{0<r<1}\{r^2(1-r)\} = \dfrac{4}{27} \approx 0.148 < 0.24$ (at $r = \frac{2}{3}$). Therefore *no GP with* $0 < r < 1$ *satisfies both conditions.*

Q11 Let terms be a, ar, ar^2 with $a > 0$. From product: $(ar)^3 = 27 \Rightarrow ar = 3 \Rightarrow a = \dfrac{3}{r}$. Sum:

$$\frac{3}{r}(1 + r + r^2) = 14 \Rightarrow 3r^2 - 11r + 3 = 0.$$

$r = \dfrac{11 \pm \sqrt{85}}{6}$, and $a = \dfrac{3}{r}$. Thus

$$(u_1, u_2, u_3) = \left(\frac{18}{11 \pm \sqrt{85}},\ 3,\ \frac{(11 \pm \sqrt{85})}{2} \right).$$

Q12 Differentiate $G(r) = \sum_{k=1}^{n} r^k = \dfrac{r(1-r^n)}{1-r}$ to get $G'(r) = \sum_{k=1}^{n} kr^{k-1} = \dfrac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$.

Q13 $3\sum k^2 - 2\sum k + \sum 1 = 3 \cdot \frac{20\cdot21\cdot41}{6} - 2 \cdot \frac{20\cdot21}{2} + 20 = 16820$.

Q14 $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1} \Rightarrow \sum_1^n = 1 - \dfrac{1}{n+1}$; for $n = 50$ the sum is $\dfrac{50}{51}$.

Q15 $\dfrac{2k+1}{k(k+1)} = \dfrac{2}{k+1} + \dfrac{1}{k} - \dfrac{1}{k+1}$, so

$$\sum_1^n \frac{2k+1}{k(k+1)} = 2\sum_1^n \frac{1}{k+1} + \sum_1^n \left(\frac{1}{k} - \frac{1}{k+1}\right) = 2(H_{n+1} - 1) + 1 - \frac{1}{n+1}.$$

Hence $= 2H_{n+1} - 1 - \dfrac{1}{n+1}$.

Q16 $\dfrac{1}{(k+1)(k+2)} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$, so the sum is $\dfrac{1}{2} - \dfrac{1}{n+2}$.

Q17 $\sum_1^n \left(\dfrac{1}{k} - \dfrac{1}{k+2}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \dfrac{3}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}$.

Q18 $S = \dfrac{1-(11)(0.8)^{10} + 10(0.8)^{11}}{(0.2)^2} \approx 16.947$.

Q19 $u_{n+1} = 1.5u_n + 4$: fixed point $u^* = \dfrac{4}{1-1.5} = -8$. $u_n = (u_0 - u^*)(1.5)^n + u^* = -8 + 10(1.5)^n$.

Q20 Fixed point $u^* = 12/(1-0.7) = 40$. With $u_1 = 5$: $u_n = (u_1 - 40)0.7^{n-1} + 40 = 40 - 35 \cdot 0.7^{n-1}$; $\lim u_n = 40$.

Q21 $u_{n+1} - u_n = 6 \Rightarrow u_n = 6n + C$. Using $u_3 = 17$: $C = -1$, so $u_n = 6n - 1$.

Q22 Solve $u_{n+1} - u^* = a(u_n - u^*)$ with $u^* = \dfrac{b}{1-a}$ to obtain $u_n = (u_0 - u^*)a^n + u^*$.

Q23 $v^* = -1$. With $v_1 = 3$: $v_n = (v_1 - v^*)2^{n-1} + v^* = (4)2^{n-1} - 1 = 2^{n+1} - 1$.

Q24 $u^* = 9/(1-0.6) = 22.5$; $u_n = 22.5 + 17.5(0.6)^n$. Need $u_n < 25 \Rightarrow (0.6)^n < 1/7 \Rightarrow n > 3.811$; least $n = 4$.

Q25 $x_{n+1} - 6 = \frac{1}{2}(x_n - 6)$; $x_n = 6 - 6\cdot 2^{-(n-1)}$. Since $x_{n+1} - x_n = \frac{1}{2}(x_n - 6) - (x_n - 6) = -\frac{1}{2}(x_n - 6) > 0$ when $x_n < 6$, the sequence is increasing and bounded above by 6; limit 6.

Q26 $\lim a_n = 3$. $a_{n+1} - a_n = -\dfrac{5}{2^{n+1}} + \dfrac{2}{3^{n+1}} < 0$ for all $n \geq 1$, hence decreasing to 3.

Q27 $u_p = u_1 + (p-1)d = 17$, $u_q = u_1 + (q-1)d = 41$. Thus $d = \dfrac{24}{q-p}$ and $u_1 = 17 - (p-1)d$.

Q28 Let $u_k = ar^{k-1} > 0$. Then $u_m u_n = a^2 r^{m+n-2} = 400$ and $u_{m+2}u_{n-2} = a^2 r^{m+n-2} = 400$ hold identically for any $r > 0$. Hence the conditions do not restrict r (if instead $u_{m+2}u_{n+2}$ was intended, positive terms would force $r = 1$).

Q29 $S_5 = \frac{5}{2}(2u_1 + 4d) = 40 \Rightarrow 2u_1 + 4d = 16$. $S_8 = 4(2u_1 + 7d) = 88 \Rightarrow 2u_1 + 7d = 22$. Subtract: $3d = 6 \Rightarrow d = 2$, $u_1 = 4$.

Q30 Using $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ and telescoping gives $4\sum k^3 = \left[(n+1)^4 - 1\right] - 6\sum k^2 - 4\sum k - n$, which simplifies to $\sum k^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

Q31 AP $5, 9, 13, \ldots$ has $d = 4$, $u_n = 4n + 1$, $S_n = \dfrac{n}{2}(5 + 4n + 1) = n(2n+3)$. Need $2n^2 + 3n > 1000$. $n > \dfrac{-3 + \sqrt{8009}}{4} \approx 21.62 \Rightarrow n = 22$.

Q32 $\dfrac{k}{(k+1)!} = \dfrac{k+1-1}{(k+1)!} = \dfrac{1}{k!} - \dfrac{1}{(k+1)!}$. Hence $\sum_{k=1}^{n} \dfrac{k}{(k+1)!} = 1 - \dfrac{1}{(n+1)!}$.

Q33 $S = \sum \dfrac{2}{3^k} = 2 \cdot \dfrac{1/3}{1 - 1/3} = 1$. Also $T = \sum k\left(\dfrac{1}{3}\right)^k = \dfrac{r}{(1-r)^2}$ with $r = \frac{1}{3}$, so $T = \dfrac{1/3}{(2/3)^2} = \dfrac{3}{4}$.

Q34 Standard induction: base $n = 0$; assume true for n, then add r^{n+1} to both sides to get the result for $n + 1$.

Challenge — Arithmetic–geometric series and optimisation (Detailed)

(a) Final volume. After n pulses:

$$V_n = \sum_{k=1}^{n} 40r^{k-1} - 5n = 40\frac{1 - r^n}{1 - r} - 5n.$$

(b) Optimal r for fixed n. Let $f(r) = \dfrac{1 - r^n}{1 - r}$ for $0 < r < 1$. Then

$$f'(r) = \frac{1 - nr^{n-1} + (n-1)r^n}{(1-r)^2}.$$

Critical points satisfy $1 - nr^{n-1} + (n-1)r^n = 0$. For $0 < r < 1$ the numerator is positive and decreases to 0 as $r \to 1^-$, so f is increasing on $(0, 1)$ and attains its supremum at $r \to 1^-$. Hence V_n is maximised by taking r as close to 1 as practical; the limiting value is

$$\lim_{r \to 1^-} V_n = 40\,n - 5n = 35n.$$

(c) Case $n = 8$. Supremum at $r \to 1^-$ gives $V_{\max} = 35 \times 8 = 280$ L. For example, $r = 0.99$ yields $V_8 = 40\frac{1-0.99^8}{0.01} - 40 \approx 269$ L, while $r = 0.999$ gives ≈ 279.4 L.

(d) Proportional leakage. With $V_{k+1} = (1 - \lambda)V_k + 40r^k$ and V_0 given, iterate to obtain

$$V_n = (1 - \lambda)^n V_0 + \sum_{j=0}^{n-1} 40r^j (1 - \lambda)^{n-1-j}.$$

This is a finite geometric–convolution sum. If $r \neq 1 - \lambda$,

$$V_n = (1-\lambda)^n V_0 + 40(1-\lambda)^n \frac{1 - \left(\dfrac{r}{1-\lambda}\right)^n}{1-\lambda-r}.$$

For n pulses, the maximum achievable final volume is $V_{\max} = 35n$ L, attained in the limit $r \to 1^-$.
For $n = 8$: $V_{\max} = 280$ L.