Integral Calculus – Answer Sheet

Q1 $\frac{2}{3}x^6 - \frac{3}{4}x^4 + x^2 - 7x + C.$

Q2 $\int 5x^{-3}\,dx = -\frac{5}{2}x^{-2},\ \int -\frac{2}{x}\,dx = -2\ln|x|,\ \int 7\,dx = 7x;$
$-\frac{5}{2x^2} - 2\ln|x| + 7x + C.$

Q3 $\frac{3}{2}e^{2x} + 4\cos x + \frac{5}{3}\sin 3x + C.$

Q4 $\int x^{1/2}dx = \frac{2}{3}x^{3/2},\ \int x^{-1/2}dx = 2x^{1/2};$
$\frac{2}{3}x^{3/2} + 2\sqrt{x} + C.$

Q5 $6\int \frac{dx}{3x+2} = 2\ln|3x+2| + C.$

Q6 $[x^3 - 2x^2 + 5x]_1^3 = 20.$

Q7 $[-\cos x + x]_0^{\pi} = \pi + 2.$

Q8 $\left[\frac{x^4}{4} - x^2\right]_{-2}^{1} = -\frac{3}{4}.$

Q9 $[e^x]_0^{\ln 5} = 4.$

Q10 $\frac{1}{2}\sin 2x\Big|_0^{\pi/2} = 0.$

Q11 $\frac{2}{3}\ln|3x+2| + C.$

Q12 $\frac{1}{4}\sin(2x^2) + C.$

Q13 $\ln(x^2+5) + C.$

Q14 Let $u = 4 - 3x$: $-\frac{2}{3}\sqrt{4-3x} + C.$

Q15 Denominator derivative $2x - 1$: $\ln(x^2 - x + 2) + C.$

Q16 $\frac{e^{2x}}{4}(2x-1) + C.$

Q17 $x\ln x - x + C.$

Q18 $u = x,\ dv = \sin 3x\,dx$: $-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C.$

Q19 $-e^{-x}(x^2 + 2x + 2) + C.$

Q20 $u = (\ln x)^2$: $x(\ln x)^2 - 2x\ln x + 2x + C.$

Q21 $\dfrac{11}{3}\ln|x-2| - \dfrac{2}{3}\ln|x+1| + C.$

Q22 Decomp. $= \dfrac{11/3}{x+2} + \dfrac{4/3}{x-1}$; $\dfrac{11}{3}\ln|x+2| + \dfrac{4}{3}\ln|x-1| + C.$

Q23 $\dfrac{7}{3}\ln|x| - \dfrac{7}{3}\ln|x+3| + C = \dfrac{7}{3}\ln\left|\dfrac{x}{x+3}\right| + C.$

Q24 Write as $\displaystyle\int\Big(\frac{2}{x-1} + \frac{5}{(x-1)^2}\Big)dx = 2\ln|x-1| - \frac{5}{x-1} + C.$

Q25 $\displaystyle\int_1^3 \Big(\frac{1}{x-2} - \frac{1}{x+2}\Big)\,dx$ is improper at $x=2$ and *diverges* (non-convergent).

Q26 Let $u = \ln x$, $dv = e^{3x}dx$: $\displaystyle\frac{e^{3x}\ln x}{3} - \frac{1}{3}\int \frac{e^{3x}}{x}\,dx = \frac{e^{3x}\ln x}{3} - \frac{1}{3}\,\mathrm{Ei}(3x) + C.$

Q27 $\ln 2 \int \frac{dx}{x} + \int \frac{\ln x}{x}dx = (\ln 2)\ln x + \frac{1}{2}(\ln x)^2 + C.$

Q28 $[x\ln x - x]_1^e = 1.$

Q29 $-\dfrac{1}{0.6}e^{-0.6x} = -\dfrac{5}{3}e^{-0.6x} + C.$

Q30 $\ln(\ln x) + C$ (for $x > 1$).

Q31 $\displaystyle\int \sin 4x\,dx = -\tfrac{1}{4}\cos 4x + C;\quad \int \cos 5x\,dx = \tfrac{1}{5}\sin 5x + C.$

Q32 $\displaystyle\tfrac{1}{2}\int \sin 2x\,dx = -\tfrac{1}{4}\cos 2x + C\ \left(= \tfrac{1}{2}\sin^2 x + C\right).$

Q33 $\tan x + C.$

Q34 $-\ln|\cos x| + C.$

Q35 $\tan x\Big|_0^{\pi/4} = 1.$

Q36 Intersections $x = -1, 2$. Area $\displaystyle\int_{-1}^{2}(x + 2 - x^2)dx = \tfrac{9}{2}.$

Q37 Intersections $0, 4$. Area $\displaystyle\int_0^4 (4x - x^2)dx = \tfrac{32}{3}.$

Q38 $f_{\text{avg}} = \dfrac{1}{\pi}\int_0^{\pi}(\sin x + 1)\,dx = 1 + \dfrac{2}{\pi}.$

Q39 $\dfrac{1}{6}\int_0^6 (6x - x^2)dx = \dfrac{36}{6} = 6.$

Q40 $\displaystyle\int_0^4 (4x - x^2)dx = \tfrac{32}{3}.$

Q41 $h = 0.5$, $x = 0, 0.5, 1, 1.5, 2$: $T \approx 0.8806$.

Q42 $h = 0.5$, $x = 0, 0.5, \ldots, 3$: $T \approx 1.24781$ (exact $= \arctan 3 \approx 1.24905$).

Q43 $h = 1$: displacement $\approx \frac{1}{2}(0 + 2.0) + \sum_{k=1}^{7} v_k = 1 + 29.5 = 30.5$ m.

Q44 Rewrite: $3/x - 4/x^2 + 1/x^3$. Integral $= 3 \ln|x| + \dfrac{4}{x} - \dfrac{1}{2x^2} + C$.

Q45 $(x^2 + 4x + 5) = (x + 2)^2 + 1$. Write $5x + 2 = \frac{5}{2}(2x + 4) - 8$.
$\displaystyle\int = \frac{5}{2} \ln\big((x + 2)^2 + 1\big) - 8 \arctan(x + 2) + C$.

Q46 $\dfrac{1}{3} \arctan\left(\dfrac{x}{3}\right) + C$.

Q47 $u = x^2$: $\frac{1}{2} \displaystyle\int_0^1 e^u \, du = \frac{1}{2}(e - 1)$.

Q48 Split at 0: $\int_{-1}^{0}(-x + 1)dx + \int_0^2 (x + 1)dx = \frac{3}{2} + 4 = \frac{11}{2}$.

Q49 $F(1) = \displaystyle\int_0^1 \frac{dx}{1 + x^2} = \arctan 1 = \frac{\pi}{4}$.

Q50 Since $\dfrac{d}{dx}\left(\frac{1}{2} \ln(1 + x^2)\right) = \dfrac{x}{1 + x^2}$,
$\displaystyle\int \frac{x}{1 + x^2} dx = \frac{1}{2} \ln(1 + x^2) + C$.

Challenge — Area bounded by a line and a parabola (Detailed)

Curves: $y = kx$ and $y = 4x - x^2$ with $k > 0$.

(a) Intersections. Solve $kx = 4x - x^2 \Rightarrow x^2 + (k - 4)x = 0 \Rightarrow x\big(x + k - 4\big) = 0$.
Hence $x = 0$ and $x = 4 - k$.

(b) Bounded region. On $(0, 4 - k)$ the difference

$$(4x - x^2) - kx = -x^2 + (4 - k)x = x\big((4 - k) - x\big)$$

is positive iff $4 - k > 0$. Thus a positive finite area exists for $0 < k < 4$ (at $k = 4$ the region collapses; $k \le 0$ is excluded by hypothesis).

(c) Area as a definite integral. With upper curve $4x - x^2$ and lower kx on $[0, 4 - k]$:

$$A(k) = \int_0^{4-k} \big(-x^2 + (4 - k)x\big)\, dx = \Big[-\tfrac{x^3}{3} + \tfrac{(4-k)x^2}{2}\Big]_0^{4-k}.$$

Let $a = 4 - k$. Then

$$A(k) = -\frac{a^3}{3} + \frac{a \cdot a^2}{2} = a^3\Big(-\tfrac{1}{3} + \tfrac{1}{2}\Big) = \frac{a^3}{6} = \frac{(4 - k)^3}{6}, \qquad 0 < k < 4.$$

(d) Maximising $A(k)$. $A(k) = \dfrac{(4-k)^3}{6}$ is strictly decreasing in k on $(0, 4)$, so the largest area on this open interval occurs as $k \to 0^+$. If $k = 0$ were permitted, $A_{\max} = \dfrac{4^3}{6} = \dfrac{64}{6} = \dfrac{32}{3}$. Therefore the supremum on $0 < k < 4$ is $32/3$, approached as $k \to 0^+$.

$$\boxed{A(k) = \frac{(4-k)^3}{6} \text{ for } 0 < k < 4, \quad A_{\max} = \frac{32}{3} \text{ at } k = 0 \text{ (or as } k \to 0^+).}$$