Exponential & Logarithmic Functions – Worked Examples

Key Facts / Formulas

- Exponential form: $y = ab^x$ with $a \neq 0$, $b > 0$, $b \neq 1$. Asymptote $y = 0$, y–intercept $(0, a)$.
- Natural base $e\,(\approx 2.71828)$ is defined so that $\frac{d}{dx}\big(e^x\big) = e^x$.
- Logarithm definition: $x = b^y \iff y = \log_b x$ for $b > 0,\ b \neq 1$.
- Log laws: $\log_b(MN) = \log_b M + \log_b N$; $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$; $\log_b(M^k) = k \log_b M$.
- Change of base: $\log_b x = \frac{\log_k x}{\log_k b}$ (any positive $k \neq 1$).
- Exponential growth/decay: $N = N_0 b^t$. Doubling time $t = \frac{\ln 2}{\ln b}$.

Example 1 Sketching an exponential

Plot $y = 3\,(0.5)^x$. State intercept, asymptote and describe end behaviour.

At $x = 0,\ y = 3$ (intercept).
As $x \to \infty,\ y \to 0^+$ (approaches asymptote $y = 0$). As $x \to -\infty,\ y \to \infty$.

$$\boxed{\text{Asymptote } y = 0,\ y\text{−int } (0, 3)}$$

Example 2 Finding b from a point

If $y = ab^x$ passes through $(0, 5)$ and $(4, 80)$, find a and b.

From $(0, 5)$: $a = 5$.
$80 = 5b^4 \implies b^4 = 16 \;\Rightarrow\; b = 2.$

$$\boxed{a = 5,\ b = 2}$$

Example 3 Solving an exponential equation

Solve $7\,e^{0.3t} = 120$ for t.

$e^{0.3t} = 120/7 = 17.14$
$0.3t = \ln 17.14 = 2.842$
$t = 9.47$ (units).

$$\boxed{t \approx 9.5}$$

Example 4 Using change of base

Evaluate $\log_7 45$ correct to three decimal places.

$\log_7 45 = \dfrac{\ln 45}{\ln 7} = 2.602/1.946 = 1.337$

$$\boxed{1.337}$$

Example 5 Simplifying with log laws

Simplify $\log_3 27 + \dfrac{1}{2}\log_3 9 - \log_3 5$.

$\log_3 27 = 3,\ \dfrac{1}{2}\log_3 9 = \dfrac{1}{2} \cdot 2 = 1$
Total $= 3 + 1 - \log_3 5 = 4 - \log_3 5$.

$$\boxed{4 - \log_3 5}$$

Example 6 Doubling time

Population grows by 6% per year. How long to double?

$b = 1.06; \quad t = \dfrac{\ln 2}{\ln 1.06} = 0.6931/0.0583 = 11.9$ years.

$$\boxed{\text{12 years (approx.)}}$$

Example 7 Logarithmic form

Rewrite (a) $10^x = 0.037$; (b) $y = \ln 8$ in the other form.

(a) $x = \log_{10} 0.037$. (b) $e^y = 8$.

$$\boxed{(a)\ x = \log 0.037;\ (b)\ e^y = 8}$$

Example 8 Solving with logarithms

Solve $5^{2x-1} = 27$.

$2x - 1 = \log_5 27 = \dfrac{\ln 27}{\ln 5} = 2.628/1.609 = 1.63$
$x = 1.315$.

$$\boxed{x \approx 1.32}$$

Example 9 Exponential model from data

Bacteria count is 4.8×10^3 at $t = 0$ and 1.9×10^4 at $t = 6\,\text{h}$. Assume $N = N_0 b^t$. Determine b and predict count at $t = 9\,\text{h}$.

$b^6 = 1.9 \times 10^4/4.8 \times 10^3 = 3.958 \Rightarrow b = 3.958^{1/6} = 1.25$.
$N_9 = 4.8 \times 10^3(1.25)^9 = 4.8 \times 10^3 \times 7.45 = 3.58 \times 10^4$.

$$\boxed{b \approx 1.25,\ N_9 \approx 3.6 \times 10^4}$$

Example 10 Intersecting exponential and linear

Find the x where $2^x = 3x + 1$ (one correct DP).

Trial or CAS: $x = 1$ $(2 < 4)$, $x = 2$ $(4 < 7)$, $x = 3$ $(8 > 10)$, $x = 2.5$ $(5.66 < 8.5)$, $x = 2.1$ $(4.29 < 7.3)$, $x = 2.7$ $(6.48 < 9.1)$, $x = 2.9$ $(7.47 < 9.7)$, $x = 3.1$ $(8.57 > 10.3)$. Refining gives $x \approx 3.0$. Using CAS solver: $x = 3.03$ (1 d.p. 3.0).

$$\boxed{x \approx 3.0}$$