Random Variables – Answer Sheet

Q1 Sum $= 1$ and all ≥ 0 (valid). $E(X) = 1.57$. $E(X^2) = 3.23$. $\mathrm{Var}(X) = 3.23 - 1.57^2 = 0.7651$.

Q2 $0.2 + q + 0.3 + p = 1 \Rightarrow p + q = 0.5$; $E(X) = -0.2 + 0 + 0.3 + 2p = 0.18 \Rightarrow p = 0.04$. Hence $q = 0.46$.

Q3 Need $a \geq 0,\ 1 - 6a \geq 0 \Rightarrow 0 \leq a \leq \frac{1}{6}$. $E(X) = 0 \cdot a + 1 \cdot (2a) + 2 \cdot (3a) + 3(1 - 6a) = 2a + 6a + 3 - 18a = 3 - 10a$.

Q4 Payoff (winnings): \$1 when $X = 1$, \$4 when $X = 2$, \$0 otherwise. Profit $Y =$ payoff $- 1$:

$$Y = \begin{cases} -1, & X = 0, 3, \\ 0, & X = 1, \\ 3, & X = 2. \end{cases}$$

$E(Y) = -1 \cdot a + 0 \cdot (2a) + 3 \cdot (3a) - 1 \cdot (1 - 6a) = (-a + 9a - 1 + 6a) = 14a - 1$.

Q5 $E(3X - 5) = 3E(X) - 5$ by linearity. $\mathrm{Var}(3X - 5) = 9\,\mathrm{Var}(X)$ (constant shift does not change variance).

Q6 $\mathrm{Var}(X) = 8.2 - (2.4)^2 = 2.44$. $\sigma = \sqrt{2.44} = 1.561$.

Q7 $\Pr(X \geq 7) = \binom{8}{7} 0.92^7 0.08 + 0.92^8 \approx 0.871$. $\Pr(X \leq 6) = 1 - 0.871 = 0.129$.

Q8 $E = 21$, $\sigma = \sqrt{13.65} = 3.693$. With C.C. $P(18 \leq X \leq 25) \approx P(17.5 \leq Y \leq 25.5)$ $z_1 = -0.948$, $z_2 = 1.218 \Rightarrow 0.888 - 0.171 \approx 0.717$.

Q9 $T \sim \mathrm{Bin}(100, 0.04)$. $\Pr(T \leq 6) = \sum_{k=0}^{6} \binom{100}{k} 0.04^k 0.96^{100-k} \approx 0.691$.

Q10 $N \sim \mathrm{NegBin}(r = 3, p = 0.78)$. $\Pr(N \leq 5) = \sum_{k=3}^{5} \binom{k-1}{2} p^3 (1-p)^{k-3} \approx 0.4746 + 0.3132 + 0.1378 = 0.9256$.

Q11 Using $(1 + x)^n$:

$$E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1 - p)^{n-k} = np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1 - p)^{(n-1)-(k-1)} = np.$$

Q12 z-scores: $z_1 = -1$, $z_2 = 1.1667$. $\Pr = \Phi(1.1667) - \Phi(-1) \approx 0.8783 - 0.1587 = 0.7196$. 90th percentile: $x = \mu + z_{0.9}\sigma = 48 + 1.2816(6) = 55.69$.

Q13 $\Phi^{-1}(0.1056) = -1.25$, $\Phi^{-1}(0.8643) = 1.10$. $(20 - \mu)/\sigma = -1.25$, $(32 - \mu)/\sigma = 1.10$. Solve: $\sigma = \dfrac{12}{2.35} = 5.106$, $\mu = 20 + 1.25\sigma = 26.383$.

Q14 $\mu = 60$, $\sigma = \sqrt{42} = 6.480$. $\Pr(X \ge 70) \approx \Pr(Y \ge 69.5) = 1 - \Phi(1.466) \approx 0.071$.

Q15 Bounds 8.8 and 11.2: $z = \pm(1.2/0.9) = \pm 1.333$. Reject proportion $= 2(1 - \Phi(1.333)) \approx 2(0.0918) = 0.184$.

Q16 $E = 6.5$, $\text{Var} = 6.75$, $\Pr(5 < X \le 8.5) = \dfrac{3.5}{9} = 0.3889$.

Q17 $k = \dfrac{3}{4}$. $F(x) = \begin{cases} 0, & x < 0, \\ \dfrac{3}{4}\left(x^2 - \dfrac{x^3}{3}\right), & 0 \le x \le 2, \\ 1, & x \ge 2, \end{cases}$ $E(X) = 1$. $E(X^2) = \dfrac{3}{4}\displaystyle\int_0^2 (2x^2 - x^3)dx = \dfrac{3}{4}\left(\dfrac{16}{3} - 4\right) = 1.2$. $\text{Var}(X) = 1.2 - 1^2 = 0.2$.

Q18 $Y = 3X - 5$. $E(Y) = 3(1) - 5 = -2$, $\text{Var}(Y) = 9(0.2) = 1.8$.
Directly: $y = 3x - 5$ maps $x \in [0, 2] \mapsto y \in [-5, 1]$. $g(y) = f\left(\frac{y+5}{3}\right) \cdot \frac{1}{3} = \dfrac{1}{4}\,\frac{y+5}{3}\left(2 - \frac{y+5}{3}\right)$ on $[-5, 1]$. Compute $\int y g(y)\, dy = -2$ and $\int y^2 g(y)\, dy = \text{Var} + E^2 = 1.8 + 4$ (confirms).

Q19 $\displaystyle\int_1^\infty \frac{c}{x^2} dx = c$. Hence $c = 1$. $\Pr(1.5 < X < 3) = \displaystyle\int_{1.5}^3 x^{-2} dx = \Big[-\tfrac{1}{x}\Big]_{1.5}^3 = \tfrac{1}{3}$. $E(X) = \int_1^\infty x \cdot x^{-2} dx = \int_1^\infty x^{-1} dx = \infty$ (does not exist); $\text{Var}(X)$ does not exist.

Q20 $E(T) = \displaystyle\int_0^\infty t\lambda e^{-\lambda t} dt = \Big[-te^{-\lambda t}\Big]_0^\infty + \int_0^\infty e^{-\lambda t} dt = 0 + \frac{1}{\lambda}$. $E(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = \dfrac{2}{\lambda^2}$; $\text{Var}(T) = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}$.

Q21 Jump at $x = 2$ is $0.5 - 0.5 = 0$ so $\Pr(X = 2) = 0$. For $0 < x < 2$, $f(x) = \dfrac{d}{dx}(x/4) = \frac{1}{4}$; for $2 < x < 3$, $f(x) = 0$; for $x > 3$, $f(x) = \dfrac{d}{dx}\left(1 - \frac{1}{x}\right) = \frac{1}{x^2}$. There is a point mass at $x = 3$ of size $F(3) - \lim_{x \to 3^-} F(x) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. $\Pr(1 < X \le 4) = F(4) - F(1) = (1 - \frac{1}{4}) - \frac{1}{4} = \frac{1}{2}$.

Q22 $Y = 2^X$ takes values $1, 2, 4, 8$ with probs $0.1, 0.3, 0.4, 0.2$. $E(Y) = 0.1 + 0.6 + 1.6 + 1.6 = 3.9$. $E(Y^2) = 0.1(1) + 0.3(4) + 0.4(16) + 0.2(64) = 20.5$. $\text{Var}(Y) = 20.5 - 3.9^2 = 5.29$.

Q23 $P(X = i) = \{1, 2, 3, 2\}/8$ so $P(X = 1, 2, 3, 4) = (1, 2, 3, 2)/8$. $E(X) = \dfrac{1 + 4 + 9 + 8}{8} = 2.75$. $E(P) = 5 - 2E(X) = 5 - 5.5 = -0.5$. $\Pr(P > 0) \Leftrightarrow X \le 2 \Rightarrow \dfrac{1+2}{8} = \dfrac{3}{8} = 0.375$.

Q24 $X + Y \sim \text{Bin}(50, 0.4)$. $\Pr(X + Y = 25) = \dbinom{50}{25}(0.4)^{25}(0.6)^{25}$.

Q25 Single-item bonus prob $p = \Pr(S \ge 85) = 1 - \Phi\left(\dfrac{85 - 74}{7}\right) = 1 - \Phi(1.571) \approx 0.058$. Let $K \sim \text{Bin}(60, 0.058)$. $\mu = 3.48$, $\sigma = \sqrt{60(0.058)(0.942)} = 1.812$. $\Pr(K \ge 8) \approx \Pr(Y \ge 7.5)$ $z = (7.5 - 3.48)/1.812 = 2.22 \Rightarrow \Pr \approx 0.013$.

Q26 (a) $P(0) = e^{-3.6} = 0.0273$. $P(\leq 2) = e^{-3.6}\left(1 + 3.6 + \dfrac{3.6^2}{2}\right) \approx 0.302$. $P(\geq 6) = 1 - \sum_{k=0}^{5} e^{-3.6}\dfrac{3.6^k}{k!} \approx 0.158$.
(b) Sum of 15 independent Pois(3.6) is Pois(54); mean $= 54$, variance $= 54$.

Q27 When p is very small or very large the binomial is highly skewed and the normal (symmetric) approximation is poor (and tails are badly estimated unless np and $n(1-p)$ are both sufficiently large). For p small with large n, use a **Poisson** approximation with $\lambda = np$; for p near 1, approximate the number of *failures* by $\text{Pois}(n(1-p))$.

Q28 The given constant makes a valid pdf only if it integrates to 1. $\displaystyle\int_0^4 \frac{3}{32}x^2 dx = \frac{3}{32}\cdot\frac{64}{3} = 2 \neq 1$, so the intended constant is almost surely $\frac{3}{64}$. Using $f(x) = \frac{3}{64}x^2$ on $[0, 4]$:

$$F(x) = \int_0^x \tfrac{3}{64}t^2 dt = \frac{x^3}{64} \quad (0 \leq x \leq 4).$$

Median m: $m^3/64 = 0.5 \Rightarrow m = \sqrt[3]{32} = 3.175$. $E(X) = \frac{3}{64}\int_0^4 x^3 dx = \frac{3}{64}\cdot\frac{256}{4} = 3$. Thus $m \approx 3.175 > E(X) = 3$ (right–skewed).

Challenge — Mixture model and classification (Detailed)

Let $S \in \{\text{A}, \text{B}\}$ with $\Pr(S = \text{A}) = 0.6$, $\Pr(S = \text{B}) = 0.4$ and

$$\Pr(D \mid S = \text{A}) = 0.02, \qquad \Pr(D \mid S = \text{B}) = 0.06.$$

(a) Marginal defect rate.

$$\Pr(D) = 0.6(0.02) + 0.4(0.06) = 0.012 + 0.024 = 0.036.$$

(b) Posterior supplier given a defect.

$$\Pr(S = \text{B} \mid D) = \frac{\Pr(D \mid S = \text{B})\Pr(S = \text{B})}{\Pr(D)} = \frac{0.06\cdot 0.4}{0.036} = \frac{0.024}{0.036} = \boxed{\tfrac{2}{3}}.$$

(c) Binomial approximation for 10 parts. Treat each part as defective with probability 0.036 independently:

$$X \sim \text{Bin}(10, 0.036), \quad \Pr(X \geq 2) = 1 - \Pr(X = 0) - \Pr(X = 1) = 1 - (0.964)^{10} - 10(0.036)(0.964)^9 \approx \boxed{0.116}.$$

(d) Exact mixture distribution. Let K be the number of parts (out of 10) drawn from supplier B. Then $K \sim \text{Bin}(10, 0.4)$. Conditional on $K = k$, defects are the sum of two independent binomials with different p:

$$X \mid K = k \;\sim\; Y + Z, \quad Y \sim \text{Bin}(10 - k, 0.02),\; Z \sim \text{Bin}(k, 0.06).$$

Hence, for $x = 0, 1, \ldots, 10$,

$$\Pr(X = x) = \sum_{k=0}^{10}\Pr(K = k)\sum_{j=0}^{x}\binom{10-k}{x-j}(0.02)^{x-j}(0.98)^{10-k-(x-j)}\binom{k}{j}(0.06)^j(0.94)^{k-j},$$

with $\Pr(K = k) = \binom{10}{k}0.4^k 0.6^{10-k}$. This is the exact law; (c) replaces it by a single $\text{Bin}(10, 0.036)$.