Random Variables – Worked Examples

Key Facts / Formulas

- A (discrete) random variable X has probability function $p(x) = \Pr(X = x)$ with $\sum p(x) = 1$ and $p(x) \geq 0$.
- Expectation (mean): $E(X) = \sum x\, p(x)$; variance $\text{Var}(X) = E(X^2) - [E(X)]^2$ where $E(X^2) = \sum x^2 p(x)$; standard deviation $\sigma = \sqrt{\text{Var}(X)}$.
- Linear transformations: if $Y = aX + b$ then $E(Y) = a\,E(X) + b$ and $\text{Var}(Y) = a^2\,\text{Var}(X)$.
- Binomial: $X \sim \text{Bin}(n, p)$ has $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $E(X) = np$, $\text{Var}(X) = np(1-p)$.
- Sums of independent binomials with same p: $X \sim \text{Bin}(n_1, p)$, $Y \sim \text{Bin}(n_2, p)$ independent $\Rightarrow X + Y \sim \text{Bin}(n_1 + n_2, p)$.
- Continuous random variable with pdf $f(x) \geq 0$, $\int f(x)\,dx = 1$ on its support. Cdf $F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$.
- For continuous X: $E(X) = \int x f(x)\,dx$, $E(X^2) = \int x^2 f(x)\,dx$, $\text{Var}(X) = E(X^2) - [E(X)]^2$.
- Uniform on $[a, b]$: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$; $E(X) = \frac{a+b}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$.
- Normal: $Z \sim N(0, 1)$; if $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma}$. Probabilities from standard normal tables or technology. For large n, $\text{Bin}(n, p)$ may be approximated by $N(np, np(1-p))$ (use continuity correction).

Example 1 Discrete table: check validity, mean and variance

A random variable X takes values $0, 1, 2, 3$ with

$$p(0) = 0.12, \quad p(1) = 0.33, \quad p(2) = 0.41, \quad p(3) = 0.14.$$

Verify this defines a probability distribution, then find $E(X)$ and $\text{Var}(X)$.

Sum $= 0.12 + 0.33 + 0.41 + 0.14 = 1.00$ and all are ≥ 0 so valid.
$E(X) = 0(0.12) + 1(0.33) + 2(0.41) + 3(0.14) = 0 + 0.33 + 0.82 + 0.42 = 1.57$.
$E(X^2) = 0 + 1(0.33) + 4(0.41) + 9(0.14) = 0.33 + 1.64 + 1.26 = 3.23$.
$\text{Var}(X) = 3.23 - (1.57)^2 = 3.23 - 2.4649 = 0.7651$.

$$\boxed{E(X) = 1.57,\ \text{Var}(X) \approx 0.765}$$

Example 2 Finding a missing probability from $E(X)$

X takes values $-1, 0, 1, 2$ with probabilities $0.2,\ q,\ 0.3,\ 0.1$ respectively. Given $E(X) = 0.18$, find q.

Sum to one: $0.2 + q + 0.3 + 0.1 = 0.6 + q \Rightarrow q = 0.4$. Check with mean:

$$E(X) = -1(0.2) + 0(q) + 1(0.3) + 2(0.1) = -0.2 + 0.3 + 0.2 = 0.30 \text{ (not } 0.18).$$

Hence adjust: include q term (value 0 contributes 0). To achieve 0.18 we must change one of the fixed probabilities; but only q was unspecified in the question stem. Therefore reinterpret: let the fourth probability be p (not 0.1). Suppose probs are $0.2,\ q,\ 0.3,\ p$ and sum to 1: $q + p = 0.5$. Mean:

$$E(X) = -0.2 + 0 + 0.3 + 2p = 0.1 + 2p = 0.18 \Rightarrow p = 0.04,\ q = 0.46.$$

$$\boxed{q = 0.46}$$

(If 0.1 was intended fixed, the given mean is inconsistent.)

Example 3 Linear transformation

For the X in Example 1, define $Y = 3X - 5$. Find $E(Y)$ and $\text{Var}(Y)$.

$E(Y) = 3E(X) - 5 = 3(1.57) - 5 = 4.71 - 5 = -0.29$.
$\text{Var}(Y) = 3^2\,\text{Var}(X) = 9(0.7651) = 6.8859$.

$$\boxed{E(Y) = -0.29,\ \text{Var}(Y) \approx 6.886}$$

Example 4 Binomial probability and cumulative

A component passes a test with probability $p = 0.92$ independently. Out of $n = 8$ components, find $\Pr(X \geq 7)$.

$X \sim \text{Bin}(8, 0.92)$. $\Pr(X \geq 7) = \Pr(X = 7) + \Pr(X = 8)$:

$$\binom{8}{7} 0.92^7 0.08^1 + 0.92^8 = 8(0.92^7)(0.08) + 0.92^8.$$

Compute $0.92^7 \approx 0.5580$, $0.92^8 \approx 0.5134$:

$$8(0.5580)(0.08) + 0.5134 = 0.3571 + 0.5134 = 0.8705.$$

$$\boxed{0.871 \text{ (approx.)}}$$

Example 5 Binomial mean and standard deviation

For $X \sim \text{Bin}(60, 0.35)$, find $E(X)$ and σ.

$E(X) = np = 60(0.35) = 21$. Variance $= np(1-p) = 60(0.35)(0.65) = 13.65$, so $\sigma = \sqrt{13.65} = 3.693$.

$$\boxed{E(X) = 21,\ \sigma \approx 3.693}$$

Example 6 Sum of independent binomials

Two inspectors check items independently. Inspector A samples 40 items with defect probability 0.04; inspector B samples 60 items with the same p. Find Pr(total defects $\leq$ 6).

$X_A \sim \text{Bin}(40, 0.04)$, $X_B \sim \text{Bin}(60, 0.04)$ independent. Then $T = X_A + X_B \sim \text{Bin}(100, 0.04)$. We need $\Pr(T \leq 6) = \sum_{k=0}^{6} \binom{100}{k} 0.04^k 0.96^{100-k}$. Evaluate with technology; result ≈ 0.691.

$$\boxed{\Pr(T \leq 6) \approx 0.691}$$

Example 7 Normal probability (z score)

Let $X \sim N(\mu = 48,\ \sigma = 6)$. Find $\Pr(42 < X < 55)$.

$Z = \dfrac{X - 48}{6}$. Bounds: $z_1 = (42 - 48)/6 = -1$, $z_2 = (55 - 48)/6 = 1.1667$.
$\Pr(42 < X < 55) = \Phi(1.1667) - \Phi(-1) = 0.8783 - 0.1587 = 0.7196$ (using tables).

$$\boxed{0.720 \text{ (approx.)}}$$

Example 8 Normal approximation to binomial with continuity correction

If $X \sim \text{Bin}(n = 200, p = 0.3)$, estimate $\Pr(X \geq 70)$ using a normal approximation.

Mean $\mu = np = 60$, variance $np(1-p) = 42$, $\sigma = \sqrt{42} = 6.480$.
Continuity: $\Pr(X \geq 70) \approx \Pr(Y \geq 69.5)$ for $Y \sim N(60, 42)$.
$z = (69.5 - 60)/6.480 = 1.466$. So probability $= 1 - \Phi(1.466) = 1 - 0.9286 = 0.0714$.

$$\boxed{0.071 \text{ (approx.)}}$$

Example 9 Uniform distribution mean and variance, and a probability

$X \sim \text{Uniform}[2, 11]$. Find $E(X)$, $\text{Var}(X)$, and $\Pr(5 < X \leq 8.5)$.

$E(X) = (2 + 11)/2 = 6.5$. $\text{Var}(X) = (11 - 2)^2/12 = 81/12 = 6.75$. Probability:

$$\Pr(5 < X \leq 8.5) = \frac{8.5 - 5}{11 - 2} = \frac{3.5}{9} = 0.3889.$$

$$\boxed{E = 6.5,\ \text{Var} = 6.75,\ \Pr = 0.3889}$$

Example 10 Continuous pdf: determine constant, cdf, mean

A continuous random variable has pdf

$$f(x) = \begin{cases} k\,x(2 - x), & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k. (b) Find $F(x)$. (c) Find $E(X)$.

(a) Total probability 1:

$$1 = k\int_0^2 x(2 - x)\,dx = k\int_0^2 (2x - x^2)\,dx = k\Big[x^2 - \tfrac{x^3}{3}\Big]_0^2 = k\Big(4 - \tfrac{8}{3}\Big) = k \cdot \tfrac{4}{3}.$$

Hence $k = \frac{3}{4}$.
(b) For $0 \le x \le 2$,

$$F(x) = \int_0^x \tfrac{3}{4}t(2-t)\,dt = \tfrac{3}{4}\left[\tfrac{t^2}{1} - \tfrac{t^3}{3}\right]_0^x = \tfrac{3}{4}\left(x^2 - \tfrac{x^3}{3}\right).$$

Also $F(x) = 0$ for $x < 0$, $F(x) = 1$ for $x \ge 2$.
(c) $E(X) = \int_0^2 x f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx.$

$$= \tfrac{3}{4}\left[\tfrac{2x^3}{3} - \tfrac{x^4}{4}\right]_0^2 = \tfrac{3}{4}\left(\tfrac{16}{3} - 4\right) = \tfrac{3}{4}\cdot\tfrac{4}{3} = 1.$$

$$\boxed{k = \tfrac{3}{4},\ F(x) = \tfrac{3}{4}\left(x^2 - \tfrac{x^3}{3}\right)\ (0 \le x \le 2),\ E(X) = 1}$$