Further Graph Transformations and Modelling – Answer Sheet

Q1 $g(x) = -\dfrac{3}{2(x-4)} + 5$; VA $x = 4$; HA $y = 5$; x–int $x = 4.3$; y–int $= 43/8$.

Q2 Domain $x \leq \frac{1}{3}$; range $[-4, \infty)$.

Q3 Vertex $(2.5, -3)$; x–ints $x = 1, 4$; y–int 2. Piecewise: $y = 2x - 8$ for $x \geq 2.5$, $y = -2x + 2$ for $x < 2.5$.

Q4 $A = -1,\ B = 3,\ h = -2,\ k = 1$ so $y = -\,f\big(3(x+2)\big) + 1$.

Q5 $p = -6,\ q = 3,\ r = -2$ so $y = -\dfrac{6}{x-3} - 2$.

Q6 With $h = 2,\ k = 1$ and point $(4, \frac{3}{2})$ gives $A = B$; one model: $A = B = 1$, $y = \dfrac{1}{x-2} + 1$.

Q7 $a = 4,\ c = 5$: $y = \dfrac{4}{(x-1)^2} + 5$.

Q8 $y = -\dfrac{4}{x+5} + 7$: VA $x = -5$; HA $y = 7$; x–int $x = -31/7$; y–int $31/5$.

Q9 $y = \dfrac{3}{(x-2)^2} - 5$: range $(-5, \infty)$; y–int $-17/4$.

Q10 $y = 4 - \sqrt{5 - 2x}$: domain $x \leq 2.5$; x–int $x = -11/2$; y–int $4 - \sqrt{5}$.

Q11 $y = \ln(3 - 2x) + 1$: domain $x < \frac{3}{2}$; VA $x = \frac{3}{2}$; x–int $= \dfrac{3 - e^{-1}}{2}$.

Q12 $y = 2|x + 1| - 5$: $y = 2x - 3$ for $x \geq -1$; $y = -2x - 7$ for $x < -1$. Vertex $(-1, -5)$; x–ints $x = 1.5, -3.5$; y–int -3.

Q13 No solution. From $(2, 2)$ we need $a\sqrt{b} = 0$, but $(5, 6)$ requires $2a\sqrt{b} = 4 \Rightarrow$ contradiction.

Q14 $0 = A/4 + B$ and $3 = -A + B \Rightarrow A = -12/5,\ B = 3/5$.

Q15 Stationary at $(1, 2)$ gives $h = 1,\ k = 2$. Using $(3, 18)$ gives $A = 2$. Hence $y = 2(x-1)^3 + 2$.

Q16 $L = 120$. From $R(0) = 40$: $a = 2$. From $R(2) = 80$: $e^{-2k} = 1/4$, so $k = \ln 2$.

Q17 $a = 4$, $b = \left(\dfrac{27}{4}\right)^{1/3}$; $y(5) \approx 612$.

Q18 $K = 2.4$; $r = \dfrac{1}{5}\ln(6.7/2.4) = 0.205$; $M(9) = 2.4e^{9r} \approx 15.24$ g.

Q19 $n = \dfrac{\ln(140/18)}{\ln(5/2)} \approx 2.24$; $k = F/v^n \approx 3.81$. Model $F \approx 3.81\, v^{2.24}$.

Q20 $a = 4.2,\ b = 1.8$; $d(4) = 4.2 + 1.8 \ln 4 \approx 6.70$ mm.

Q21 With HA $y = 1$: $y = \dfrac{p}{x-q} + 1$. From $(2, 10)$ and $(6, 2)$: $q = 1.5$, $p = 4.5$. So $y = \dfrac{4.5}{x - 1.5} + 1$.

Q22 $x^2 - 4x \geq 0$ for $x \leq 0$ or $x \geq 4$. Thus

$$y = \begin{cases} x^2 - 4x, & x \leq 0 \text{ or } x \geq 4, \\ -x^2 + 4x, & 0 < x < 4. \end{cases}$$

x–ints $x = 0, 4$.

Q23 $|2x - 5| - 3 \leq 1 \Rightarrow |2x - 5| \leq 4 \Rightarrow 0.5 \leq x \leq 4.5$.

Q24 Vertex at $x = 2$ gives $c = -2m$ and $k = -3$. Using $(0, 1)$: $|c| - 3 = 1 \Rightarrow |2m| = 4$. Hence $m = 2,\ c = -4$ or $m = -2,\ c = 4$ (both valid).

Q25 Domain $(-\infty, 7]$; range $(-\infty, 3]$.

Q26 Range of f is $[2, \infty) \Rightarrow$ range of h is $-3[2, \infty) + 5 = (-\infty, -1]$.

Q27 Require input to f satisfy $3 - 2x > 5$ (and > 5 avoids the original restriction boundary): $x < -1$. Hence domain $(-\infty, -1)$.

Q28 $T(t) = 22 + 58e^{-kt}$, $40 = 22 + 58e^{-10k} \Rightarrow e^{-10k} = 18/58$. Thus $k = 0.1170$; $T(20) = 22 + 58(18/58)^2 \approx 27.6°$C.

Q29 $R(t) = \dfrac{120000}{1 + ae^{-kt}}$. $R(0) = 5000 \Rightarrow a = 19$. $R(3) = 30000 \Rightarrow e^{-3k} = 7/57$, so $k = 0.699$. $R(6) = \dfrac{120000}{1 + 19(7/57)^2} \approx 7.77 \times 10^4$.

Q30 $C(t) = \dfrac{A}{t+q} + 6$. $C(0) = 42 \Rightarrow A/q = 36$. $C(5) = 18 \Rightarrow A/(5+q) = 12$. Solve: $q = 2.5$, $A = 90$.

Challenge – Selecting a model and linearising (Detailed)

(a) Linearisation.

- Log model $y = a + b \ln x$: let $X = \ln x$, $Y = y$. Then $Y = a + bX$.
- Reciprocal approach $y = c - \dfrac{d}{x}$: let $X = \dfrac{1}{x}$, $Y = y$. Then $Y = c - dX$.

(b) Hand estimates from $(1, 4.1)$ and $(12, 9.6)$; check at $(3, 6.6)$.

- Log: $b = \dfrac{9.6 - 4.1}{\ln 12 - \ln 1} = \dfrac{5.5}{\ln 12} = 2.213$; $a = 4.1$.
  Check $x = 3$: $y \approx 4.1 + 2.213 \ln 3 = 6.53$ (close to 6.6).

- Reciprocal: slope using $X = 1/x$: $X_1 = 1,\ X_2 = \frac{1}{12}$.
  d from $m = \dfrac{9.6 - 4.1}{\frac{1}{12} - 1} = -6.00 \Rightarrow d = 6.00$, c from $4.1 = c - 6(1)$ gives $c = 10.1$.
  Check $x = 3$ ($X = \frac{1}{3}$): $y = 10.1 - 6(\frac{1}{3}) = 8.10$ (not close).

(c) Prediction at $x = 20$ and comparison to target 10.

$$\text{Log: } y(20) = 4.1 + 2.213 \ln 20 \approx 10.73, \quad |10.73 - 10| = 0.73.$$
$$\text{Reciprocal: } y(20) = 10.1 - \frac{6}{20} = 9.80, \quad |9.80 - 10| = 0.20.$$

$$\boxed{\text{Reciprocal model gives smaller error at } x = 20.}$$

(d) Plausibility. The reciprocal model approaches the finite ceiling $y \to c \approx 10.1$ as $x \to \infty$, matching a known upper bound near 10. The logarithmic model grows without bound (albeit slowly), conflicting with a hard cap. Hence the reciprocal model is more physically plausible for a saturating device response.