Integral Calculus – Worked Examples

Key Facts / Formulas

- Antiderivative rules: $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$; $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$; $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$; $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$; $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$.
- Substitution: if $u = g(x)$, then $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$.
- Integration by parts: $\displaystyle\int u\,dv = uv - \int v\,du$.
- Partial fractions (distinct linear factors): $\displaystyle\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$.
- Definite integral: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F of f.
- Area between curves $y = f(x)$ above $y = g(x)$ on $[a,b]$: $\displaystyle\int_a^b \big(f(x) - g(x)\big)\,dx$.
- Average value on $[a,b]$: $\displaystyle f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.
- Trapezoidal rule with step $h = \frac{b-a}{n}$: $\displaystyle\int_a^b f(x)\,dx \approx \frac{h}{2}\Big(f_0 + 2\sum_{k=1}^{n-1} f_k + f_n\Big)$.

Example 1 Basic antiderivative of a polynomial

Find $\displaystyle\int \big(4x^5 - 3x^3 + 2x - 7\big)\,dx$.

Integrate term by term:

$$\int 4x^5\,dx = \tfrac{4}{6}x^6 = \tfrac{2}{3}x^6, \quad \int(-3x^3)\,dx = -\tfrac{3}{4}x^4, \quad \int 2x\,dx = x^2, \quad \int(-7)\,dx = -7x.$$

Hence

$$\int \big(4x^5 - 3x^3 + 2x - 7\big)\,dx = \tfrac{2}{3}x^6 - \tfrac{3}{4}x^4 + x^2 - 7x + C.$$

$$\boxed{\tfrac{2}{3}x^6 - \tfrac{3}{4}x^4 + x^2 - 7x + C}$$

Example 2 Definite integral of a polynomial

Evaluate $\displaystyle\int_1^3 \left(3x^2 - 4x + 5\right) dx$.

An antiderivative is $x^3 - 2x^2 + 5x$. Thus

$$\Big[x^3 - 2x^2 + 5x\Big]_1^3 = (27 - 18 + 15) - (1 - 2 + 5) = (24) - (4) = 20.$$

$$\boxed{20}$$

Example 3 Substitution producing a logarithm

Find $\displaystyle\int \frac{6x+4}{(3x+2)^2}\, dx$.

Let $u = 3x + 2$, so $du = 3\, dx$ and $dx = \frac{du}{3}$. Note $6x + 4 = 2(3x + 2) = 2u$. Then

$$\int \frac{6x+4}{(3x+2)^2}\, dx = \int \frac{2u}{u^2} \cdot \frac{du}{3} = \frac{2}{3} \int u^{-1}\, du = \frac{2}{3} \ln|u| + C = \frac{2}{3} \ln|3x+2| + C.$$

$$\boxed{\tfrac{2}{3} \ln|3x+2| + C}$$

Example 4 Substitution with a quadratic inside a trig function

Find $\displaystyle\int x \cos(2x^2)\, dx$.

Let $u = 2x^2$, $du = 4x\, dx$, so $x\, dx = \frac{du}{4}$. Then

$$\int x \cos(2x^2)\, dx = \frac{1}{4} \int \cos u\, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(2x^2) + C.$$

$$\boxed{\tfrac{1}{4} \sin(2x^2) + C}$$

Example 5 Integration by parts: polynomial times exponential

Evaluate $\displaystyle\int xe^{2x}\, dx$.

Let $u = x$ $(du = dx)$ and $dv = e^{2x}\, dx$ $(v = \frac{1}{2}e^{2x})$. Then

$$\int xe^{2x}\, dx = uv - \int v\, du = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\, dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C = \frac{e^{2x}}{4}(2x - 1) + C.$$

$$\boxed{\tfrac{e^{2x}}{4}(2x - 1) + C}$$

Example 6 Integration by parts: $\displaystyle\int \ln x\, dx$

Show that $\displaystyle\int \ln x\, dx = x \ln x - x + C$.

Let $u = \ln x$ ($du = \frac{1}{x}dx$) and $dv = dx$ ($v = x$). Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$$

$$\boxed{x \ln x - x + C}$$

Example 7 Partial fractions with distinct linear factors

Find $\displaystyle\int \frac{3x+5}{x^2-x-2} \, dx$.

Factor the denominator: $x^2 - x - 2 = (x-2)(x+1)$. Write

$$\frac{3x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

So $3x + 5 = A(x+1) + B(x-2) = (A+B)x + (A-2B)$. Equate coefficients:

$$A + B = 3, \quad A - 2B = 5 \;\Rightarrow\; 3B = -2 \Rightarrow B = -\tfrac{2}{3}, \quad A = 3 - B = \tfrac{11}{3}.$$

Hence

$$\int \frac{3x+5}{x^2-x-2} \, dx = \frac{11}{3} \int \frac{dx}{x-2} - \frac{2}{3} \int \frac{dx}{x+1} = \frac{11}{3} \ln|x-2| - \frac{2}{3} \ln|x+1| + C.$$

$$\boxed{\tfrac{11}{3} \ln|x-2| - \tfrac{2}{3} \ln|x+1| + C}$$

Example 8 Definite area between two curves

Find the area enclosed by $y = x + 2$ and $y = x^2$ between their points of intersection.

Solve $x + 2 = x^2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$, so $x = -1, 2$. On $[-1, 2]$, $x + 2 \geq x^2$. Thus

$$\text{Area} = \int_{-1}^{2} \big((x+2) - x^2\big) \, dx = \left[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\right]_{-1}^{2}.$$

At 2: $\frac{1}{2}(4) + 4 - \frac{1}{3}(8) = 2 + 4 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}$.
At -1: $\frac{1}{2}(1) - 2 - \left(-\frac{1}{3}\right) = \frac{1}{2} - 2 + \frac{1}{3} = -\frac{7}{6}$. Therefore

$$\text{Area} = \tfrac{10}{3} - \left(-\tfrac{7}{6}\right) = \tfrac{20}{6} + \tfrac{7}{6} = \tfrac{27}{6} = \tfrac{9}{2}.$$

$$\boxed{\tfrac{9}{2}}$$

Example 9 Average value on an interval

Find the average value of $f(x) = \sin x + 1$ on $[0, \pi]$.

$$f_{\text{avg}} = \frac{1}{\pi - 0} \int_0^{\pi} (\sin x + 1) \, dx = \frac{1}{\pi} \Big[-\cos x + x \Big]_0^{\pi} = \frac{1}{\pi} \big(-\cos \pi + \pi - (-\cos 0 + 0) \big).$$

Since $\cos \pi = -1$ and $\cos 0 = 1$,

$$f_{\text{avg}} = \frac{1}{\pi} \big(1 + \pi - (-1)\big) = \frac{\pi + 2}{\pi} = 1 + \frac{2}{\pi}.$$

$$\boxed{1 + \tfrac{2}{\pi}}$$

Example 10 Trapezoidal rule approximation

Approximate $\int_0^2 e^{-x^2}\,dx$ using the trapezoidal rule with $n = 4$ equal subintervals.

Step $h = \frac{2-0}{4} = 0.5$. Points $x_k = 0, 0.5, 1.0, 1.5, 2.0$ and

$$f_0 = e^0 = 1, \quad f_1 = e^{-0.25} = 0.7788008,$$
$$f_2 = e^{-1} = 0.3678794, \quad f_3 = e^{-2.25} = 0.105399, \quad f_4 = e^{-4} = 0.018315.$$

Trapezoidal:

$$T = \frac{h}{2}\Big(f_0 + 2(f_1 + f_2 + f_3) + f_4\Big) = 0.25\Big(1 + 2(0.7788008 + 0.3678794 + 0.105399) + 0.018315\Big).$$

Inside sum $= 1 + 2(1.2520792) + 0.018315 = 1 + 2.5041584 + 0.018315 = 3.5224734$. Hence

$$T = 0.25 \times 3.5224734 = 0.880618.$$

(Exact value ≈ 0.882081; error $\approx 1.46 \times 10^{-3}$.)

$$\boxed{0.8806 \text{ (to 4 d.p.)}}$$