Differential Calculus – Answer Sheet

Q1 $f'(x) = 20x^4 - 9x^2 + 2.$

Q2 $f'(x) = -\frac{5}{2}x^{-3/2} + 6x^{-3}.$

Q3 $y' = 4(x^2 + 3x - 1)^3(2x + 3).$

Q4 $y' = -\frac{2}{\sqrt{9 - 4x}}.$

Q5 $f'(2) = 8; \quad y - f(2) = 8(x - 2)$ and $f(2) = 0 \Rightarrow \boxed{y = 8x - 16}.$

Q6 $y' = e^{3x}(6x^2 + 4x - 15).$

Q7 $y' = 2x \sin x + (x^2 + 1) \cos x.$

Q8 $y' = \frac{-3x^2 + 8x - 3}{x^4}.$

Q9 $y' = \frac{x \cos x - \sin x}{x^2}.$

Q10 $y' = \frac{(x - 1)(-x + 5)}{(x + 1)^4}.$

Q11 $y' = 20 \sin^4(4x) \cos(4x).$

Q12 $y' = -6 \cos^2(2x - 1) \sin(2x - 1).$

Q13 $y' = \frac{-2(1 + 3x)}{5 - 2x - 3x^2}.$

Q14 $y' = e^{x^2 - 3x}(2x - 3).$

Q15 $y' = 3\sqrt{2x - 1}.$

Q16 $y' = e^{2x}\left(2 \ln x + \frac{1}{x}\right).$

Q17 $y' = \frac{2x}{x^2 + 4} - \frac{1}{2x}.$

Q18 $y' = 7^x e^{-3x}(\ln 7 - 3).$

Q19 $y' = \frac{2x}{x^2 + 1}, \quad$ tangent at $x = 2$: $y - \ln 5 = \frac{4}{5}(x - 2).$

Q20 $y' = \sec^2 x + \sec x \tan x$.

Q21 $y' = 3\cos(3x)\cos(2x) - 2\sin(3x)\sin(2x)$.

Q22 $y' = 10\ \tan(5x-2)\sec^2(5x-2)$.

Q23 $\cot x = \dfrac{\cos x}{\sin x} \Rightarrow \dfrac{d}{dx}\cot x = \dfrac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = -\csc^2 x$.

Q24 $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$.

Q25 $\cos(x+y)\big(1+y'\big) = y + xy' \Rightarrow y' = \dfrac{y-\cos(x+y)}{\cos(x+y)-x}$.

Q26 $3x^2 + 3y^2y' = 6y + 6xy' \Rightarrow y' = \dfrac{2y-x^2}{y^2-2x}$; at $(2,2)$: $y' = 0$.

Q27 $2x - 3(y+xy') + 2yy' = 0 \Rightarrow (-3x+2y)y' = 3y-2x$;
$y'(1,1) = -1$, tangent: $\boxed{y = -x+2}$.

Q28 $\dfrac{dy}{dx} = \dfrac{3(t^2-1)}{2t}, \quad \left.\dfrac{d^2y}{dx^2}\right|_{t=1} = \dfrac{3}{2}$.

Q29 $dy/dx = \dfrac{-\sin t}{\cos t} = -\tan t$; at $t = \pi/4$: slope -1.

Q30 $dy/dx = \dfrac{(t+1)e^t}{e^t} = t+1$; at $t=0$: point $(1,0)$, tangent $y = x-1$.

Q31 $f'(x) = 4x^3 - 8x$, $f''(x) = 12x^2 - 8$; $f''(0) = -8 < 0$ so local maximum at $(0,3)$.

Q32 $y' = 3(x-1)(x-3)$, $y'' = 6x - 12$. At $x=1$, $y'' < 0 \Rightarrow$ local max $(1,8)$; at $x=3$, $y'' > 0 \Rightarrow$ local min $(3,4)$.

Q33 $y' = 3x^2 - 3$, $y'' = 6x$; point of inflection where y'' changes sign: $x = 0$ (at $(0,0)$).

Q34 Tangent $y - \ln 5 = \dfrac{4}{5}(x-2)$; Normal $y - \ln 5 = -\dfrac{5}{4}(x-2)$.

Q35 $y' = -\dfrac{3}{(x-2)^2}$; at $x=3$: gradient -3, point $(3,4)$.
Tangent $y - 4 = -3(x-3) \Rightarrow y = -3x+13$; Normal $y - 4 = \frac{1}{3}(x-3) \Rightarrow y = \frac{1}{3}x + 3$.

Q36 $y'(1) = 2 + \dfrac{1}{2} = \dfrac{5}{2}$; tangent $y - 3 = \dfrac{5}{2}(x-1)$.

Q37 $y' = \dfrac{(x^2+1)^5\sqrt{3x-1}}{e^{2x}}\left(\dfrac{10x}{x^2+1} + \dfrac{3}{2(3x-1)} - 2\right)$.

Q38 Let $y = (x^2+3x+2)^x$. $\ln y = x\ln(x^2+3x+2)$.
$\dfrac{y'}{y} = \ln(x^2+3x+2) + x\,\dfrac{2x+3}{x^2+3x+2}$, so

$$y' = (x^2+3x+2)^x\left[\ln(x^2+3x+2) + \frac{x(2x+3)}{x^2+3x+2}\right].$$

Q39 $y' = \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow x = \pm 1.$
$x=1$: max, $y=\frac{1}{2}$; $\quad x=-1$: min, $y=-\frac{1}{2}$.

Q40 $y' = e^{-x}(2x-x^2) = 0 \Rightarrow x = 0, 2$; maximum at $x=2$, $y=4e^{-2}$.

Q41 $y = \arctan(\sqrt{x})$: $y' = \dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}.$

Q42 $y' = \cot x$; $y'' = -\csc^2 x$.

Q43 $y = x^{\sin x}$: $\ln y = \sin x \ln x \Rightarrow \dfrac{y'}{y} = \dfrac{\sin x}{x} + \cos x \ln x.$

Challenge — Envelope and common tangent (Detailed)

Family $y = \dfrac{k}{x} + x$ with $k > 0$.

(a) Stationary point

$y' = 1 - \dfrac{k}{x^2} = 0 \Rightarrow x = \sqrt{k}$ (take $x>0$ as $k>0$).
$y'' = \dfrac{2k}{x^3} > 0$ at $x = \sqrt{k}$, hence a local minimum at

$$\boxed{(\sqrt{k},\ 2\sqrt{k})}.$$

(b) Tangent at $x = a$

Slope $m = y'(a) = 1 - \dfrac{k}{a^2}$. Passing through $\left(a,\ \dfrac{k}{a}+a\right)$ gives

$$c = \left(\frac{k}{a}+a\right) - ma = \frac{2k}{a}.$$

Thus the correct intercept is $c = \dfrac{2k}{a}$ *(the extra* $+a$ *does not appear).*

(c) "Envelope" via elimination

From $m = 1 - \dfrac{k}{x^2}$ we have $k = x^2(1-m)$. Sub into $y = \dfrac{k}{x} + x$:

$$y = x(1-m) + x = 2x - xm.$$

Writing $m = y'$ gives the first–order Lagrange equation

$$x\,y' + y = 2x \quad\Longleftrightarrow\quad \frac{d}{dx}(xy) = 2x.$$

Integrating: $xy = x^2 + C \quad \Rightarrow \quad y = \dfrac{x^2 + C}{x} = x + \dfrac{C}{x}$, which reproduces the original family (no non-trivial singular solution).
Hence there is *no genuine envelope curve* distinct from the family. The locus of stationary points is $x = \sqrt{k},\ y = 2\sqrt{k}$, i.e.

$$\boxed{y = 2x \quad (x > 0),}$$

which is the locus of minima; it touches no member (their slopes there are 0 while the line has slope 2), so it is not an envelope in the strict tangency sense.

(d) Common tangent with the locus $y = 2x$

A common tangent to $y = 2x$ has slope 2. A member has slope $y' = 1 - \dfrac{k}{x^2}$. Setting $1 - \dfrac{k}{x^2} = 2$ gives $k = -x^2$, impossible for $k > 0$.

$$\boxed{\text{No common tangent of slope 2 exists for } k > 0;\ \text{hence there is no common tangent with } y = 2x.}$$