Sequences and Series - Worked Examples

Key Facts / Formulas

- Arithmetic progression (AP): $u_n = u_1 + (n-1)d$. Sum $S_n = \frac{n}{2}\big(2u_1 + (n-1)d\big) = \frac{n}{2}(u_1 + u_n)$.
- Geometric progression (GP): $u_n = u_1 r^{\,n-1}$. Finite sum $S_n = u_1 \frac{1 - r^{\,n}}{1 - r}$ for $r \neq 1$. Infinite sum (if $|r| < 1$): $S_\infty = \frac{u_1}{1 - r}$.
- Sigma facts: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.
- Telescoping example: $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$.
- Linear first order recurrence $u_{n+1} = a u_n + b$ has closed form $u_n = C\,a^{\,n} + u^*$ where u^* solves $u^* = a u^* + b$ (i.e. $u^* = \frac{b}{1-a}$ when $a \neq 1$) and $C = u_0 - u^*$.
- Weighted geometric sums: $\displaystyle\sum_{k=1}^{n} k r^{k-1} = \frac{1 - (n+1)r^n + n r^{n+1}}{(1-r)^2}$ for $r \neq 1$.

Example 1 AP terms and sum

An AP has $u_1 = 7$ and common difference $d = 3$. Find u_{20} and S_{20}.

$u_{20} = 7 + (20-1)\cdot 3 = 7 + 57 = 64.$

$S_{20} = \frac{20}{2}(u_1 + u_{20}) = 10(7 + 64) = 710.$

$$\boxed{u_{20} = 64, \quad S_{20} = 710}$$

Example 2 AP from a given sum formula

An AP has partial sums $S_n = 2n^2 - 5n$. Find u_n, then u_1 and d.

$u_n = S_n - S_{n-1} = (2n^2 - 5n) - \big(2(n-1)^2 - 5(n-1)\big)$

$= 2n^2 - 5n - \big(2n^2 - 4n + 2 - 5n + 5\big) = 2n^2 - 5n - (2n^2 - 9n + 7) = 4n - 7.$

Thus $u_n = 4n - 7 = u_1 + (n-1)d \Rightarrow u_1 = 4(1) - 7 = -3$, $d = 4$.

$$\boxed{u_n = 4n - 7, \quad u_1 = -3, \quad d = 4}$$

Example 3 GP term and sum

A GP has $u_1 = 250$ and ratio $r = 0.8$. Find u_8 and S_8.

$u_8 = 250(0.8)^7 = 250(0.2097152) = 52.4288.$
$S_8 = 250\dfrac{1-0.8^8}{1-0.8} = 250\dfrac{1-0.16777216}{0.2} = 250 \cdot 4.1611392 = 1040.2848.$

$$\boxed{u_8 \approx 52.43, \quad S_8 \approx 1040.28}$$

Example 4 Infinite GP and threshold index

For the GP with $u_1 = 12$ and $r = \dfrac{2}{3}$, find S_∞. Determine the least n such that $S_n > 17$.

$S_\infty = \dfrac{12}{1-2/3} = 36$. Also $S_n = 12\dfrac{1-(2/3)^n}{1-2/3} = 36\big(1-(2/3)^n\big)$.

We need $36\big(1-(2/3)^n\big) > 17 \Rightarrow 1-(2/3)^n > \dfrac{17}{36} \Rightarrow (2/3)^n < \dfrac{19}{36}$.

Take logs: $n\ln(2/3) < \ln(19/36)$; since $\ln(2/3) < 0$, inequality reverses: $n > \dfrac{\ln(19/36)}{\ln(2/3)} \approx \dfrac{-0.6391}{-0.4055} = 1.576$. Smallest integer $n = 2$.

$$\boxed{S_\infty = 36, \quad \text{least } n = 2}$$

Example 5 Find GP ratio from two terms

In a GP, $u_3 = 45$ and $u_6 = 360$. Find u_1 and r.

$u_3 = u_1 r^2 = 45$, $u_6 = u_1 r^5 = 360$. Divide: $\dfrac{u_6}{u_3} = r^3 = \dfrac{360}{45} = 8$, so $r = 2$.

Then $u_1 = \dfrac{u_3}{r^2} = \dfrac{45}{4} = 11.25$.

$$\boxed{u_1 = 11.25, \quad r = 2}$$

Example 6 Sigma of a quadratic

Evaluate $\displaystyle\sum_{k=1}^{20}\big(3k^2 - 2k + 1\big)$.

Use identities: $\sum k^2 = \dfrac{n(n+1)(2n+1)}{6}$, $\sum k = \dfrac{n(n+1)}{2}$, $\sum 1 = n$. For $n = 20$:

$$3\sum k^2 - 2\sum k + \sum 1 = 3 \cdot \frac{20 \cdot 21 \cdot 41}{6} - 2 \cdot \frac{20 \cdot 21}{2} + 20.$$

Compute: $\dfrac{20 \cdot 21 \cdot 41}{6} = 20 \cdot 7 \cdot 41 = 5740$, times 3 gives $17\,220$. Next term $= 20 \cdot 21 = 420$. Hence $17\,220 - 420 + 20 = 16\,820$.

$$\boxed{16\,820}$$

Example 7 Telescoping sum

Find $\displaystyle\sum_{k=1}^{50}\frac{1}{k(k+1)}$.

Partial fractions: $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Thus

$$\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{51} = \frac{50}{51}.$$

$$\boxed{\frac{50}{51}}$$

Example 8 First order linear recurrence

Solve $u_{n+1} = 1.5\,u_n + 4$ with $u_0 = 2$.

Steady state u^* satisfies $u^* = 1.5u^* + 4 \Rightarrow -0.5u^* = 4 \Rightarrow u^* = -8$.
General form $u_n = C(1.5)^n + u^*$. With $u_0 = 2$: $2 = C - 8 \Rightarrow C = 10$. Hence

$$u_n = -8 + 10(1.5)^n.$$

$$\boxed{u_n = -8 + 10\left(\tfrac{3}{2}\right)^n}$$

Example 9 Arithmetic–geometric type sum

Compute $S = \displaystyle\sum_{k=1}^{10} k(0.8)^{k-1}$.

Use formula $\displaystyle\sum_{k=1}^{n} kr^{k-1} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$ for $r \neq 1$. With $r = 0.8$, $n = 10$:

$$S = \frac{1 - 11(0.8)^{10} + 10(0.8)^{11}}{(0.2)^2}.$$

$(0.8)^{10} = 0.1073741824$, $(0.8)^{11} = 0.08589934592$. Numerator $= 1 - 11(0.1073741824) + 10(0.08589934592)$
$= 1 - 1.1811160064 + 0.8589934592 = 0.6778774528$. Denominator $= 0.04$. So $S = 16.94693632 \approx 16.947$.

$$\boxed{16.947 \text{ (to 3 d.p.)}}$$

Example 10 Induction: sum of first n odd numbers

Prove $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Base $n = 1$: LHS $= 1$, RHS $= 1^2$ true. Assume true for $n = k$: $\sum_{j=1}^{k}(2j-1) = k^2$. Then for $k+1$:

$$\sum_{j=1}^{k+1}(2j-1) = k^2 + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2.$$

Thus holds for all $n \in \mathbb{N}$ by induction.

$$\boxed{\sum_{j=1}^{n}(2j-1) = n^2}$$