Working with Functions – Worked Examples

Key Facts / Formulas

- **Index laws** – e.g. $a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, $a^{-n} = 1/a^n$
- **Quadratic discriminant**: $\Delta = b^2 - 4ac$; $\Delta > 0$ two real roots, $\Delta = 0$ equal roots, $\Delta < 0$ no real roots
- A **function** assigns each x exactly one y; a relation need not be single-valued
- **Vertical-line test**: a graph is a function iff every vertical line meets it once at most
- **Domain / range**: sets of permissible x and resulting y values
- **Even / odd tests**: $f(-x) = f(x)$ (even), $f(-x) = -f(x)$ (odd)
- **Composite function**: $(g \circ f)(x) = g\big(f(x)\big)$
- **Direct / inverse variation**: $y = kx^n$ (direct), $xy = k$ (inverse) for constant k
- **Circle equation**: $(x-h)^2 + (y-k)^2 = r^2$ centre (h, k), radius r
- **Piecewise functions**: rule changes on different domain intervals

Example 1 Simplifying with index laws

Simplify $3a^{-2}b^4 \times 6a^3b^{-1}$.

Combine coefficients: $3 \times 6 = 18$; $a^{-2}a^3 = a^1 = a$; $b^4 b^{-1} = b^3$.

$$\boxed{18ab^3}$$

Example 2 Algebraic fraction

Simplify $\dfrac{4x^2 - 9}{2x - 3}$.

Factor numerator as difference of squares: $(2x+3)(2x-3)$; Cancel common $(2x-3)$.

$$\boxed{2x + 3}$$

Example 3 Roots via discriminant

How many real roots does $5t^2 - 4t + 3 = 0$ have?

$\Delta = b^2 - 4ac = (-4)^2 - 4(5)(3) = 16 - 60 = -44 < 0$.

$$\boxed{\text{No real roots}}$$

Example 4 Function or not?

Does the graph of $x^2 + y^2 = 9$ represent y as a function of x?

Vertical line $x = 0$ meets circle at $(0, 3)$ and $(0, -3)$ two y values.

$$\boxed{\text{Not a function}}$$

Example 5 Domain & range

For $f(x) = \sqrt{5 - 2x}$ find domain and range.

Inside root ≥ 0: $5 - 2x \geq 0 \Rightarrow x \leq 2.5$. Domain $(-\infty, 2.5]$. $y \geq 0$ so range $[0, \infty)$.

$$\boxed{\text{Dom } (-\infty, 2.5],\ \text{Ran } [0, \infty)}$$

Example 6 Even or odd?

Determine if $g(x) = x^3 - 2x$ is even, odd or neither.

$g(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -\big(x^3 - 2x\big) = -g(x)$.

$$\boxed{\text{Odd}}$$

Example 7 Composite function value

Let $f(x) = 3x - 5$, $g(x) = x^2 + 1$. Find $(g \circ f)(2)$.

$f(2) = 3(2) - 5 = 1$; $g(1) = 1^2 + 1 = 2$.

$$\boxed{2}$$

Example 8 Equation of a line

Find the line through $(4, -1)$ and $(-2, 5)$.

Gradient $m = \dfrac{5 - (-1)}{-2 - 4} = \dfrac{6}{-6} = -1$. Point–gradient: $y + 1 = -1(x - 4) \Rightarrow y = -x + 3$.

$$\boxed{y = -x + 3}$$

Example 9 Direct variation model

Grape mass m varies directly with volume V; $m = 250$ g when $V = 190$ cm^3. Find m when $V = 220$ cm^3.

$m = kV \to k = \frac{250}{190} = 1.316$. $m = 1.316(220) = 289.5$ g.

$$\boxed{2.90 \times 10^2 \text{ g}}$$

Example 10 Circle from centre and radius

Centre (−3, 2), radius 5. Write the equation.

$(x+3)^2+(y-2)^2=25.$

$$\boxed{(x+3)^2+(y-2)^2=25}$$