

Trigonometry – Worked Examples

Key Facts / Formulas

- **Right-triangle ratios:** $\sin \theta = \frac{\text{opp}}{\text{hyp}}, \cos \theta = \frac{\text{adj}}{\text{hyp}}, \tan \theta = \frac{\text{opp}}{\text{adj}}.$
- **Reciprocal ratios:** $\sec \theta = \frac{1}{\cos \theta}, \csc \theta = \frac{1}{\sin \theta}, \cot \theta = \frac{1}{\tan \theta}.$
- **Pythagorean identity:** $\sin^2 \theta + \cos^2 \theta = 1.$
- **Complementary identities:** $\sin(90^\circ - \theta) = \cos \theta, \tan(90^\circ - \theta) = \cot \theta, \text{ etc.}$
- **Periodicity / π -shift identities:**

$$\sin(\theta + \pi) = -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta, \quad \tan(\theta + \pi) = \tan \theta,$$

$$\csc(\theta + \pi) = -\csc \theta, \quad \sec(\theta + \pi) = -\sec \theta, \quad \cot(\theta + \pi) = \cot \theta.$$

- **Radians-degrees:** $180^\circ = \pi \text{ rad}; \ell = r\theta \text{ (arc) and } A = \frac{1}{2}r^2\theta \text{ (sector) when } \theta \text{ in rad.}$
- **Sine rule:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$ **Cosine rule:** $c^2 = a^2 + b^2 - 2ab \cos C.$
- **Exact ratios:** $\sin 30^\circ = \frac{1}{2}, \cos 45^\circ = \frac{\sqrt{2}}{2}, \tan 60^\circ = \sqrt{3}.$

Example 1 Exact ratios using a special triangle

Find $\sin 45^\circ, \cos 45^\circ, \tan 45^\circ$ without a calculator.

Using an isosceles right triangle with legs 1: $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \tan 45^\circ = 1.$

$\sin = \cos = \frac{\sqrt{2}}{2}, \tan = 1$

Example 2 Degrees to radians

Convert 135° to radians and $-\frac{\pi}{6}$ to degrees.

$$135^\circ = \frac{135\pi}{180} = \frac{3\pi}{4} \text{ rad}; \quad -\frac{\pi}{6} \text{ rad} = -30^\circ.$$

$\frac{3\pi}{4} \text{ rad}; \quad -30^\circ$

Example 3 Right-triangle problem with bearing

A hiker walks 1.8km on a bearing 040° then 2.5km due east. How far and on what bearing is she from the start?

Resolve first leg: $x = 1.8 \sin 40^\circ = 1.16\text{km}$ east, $y = 1.8 \cos 40^\circ = 1.38\text{km}$ north. Total east $= 1.16 + 2.5 = 3.66\text{km}$; north $= 1.38\text{km}$. Distance $d = \sqrt{3.66^2 + 1.38^2} = 3.91\text{km}$. Bearing $\theta = \tan^{-1}\left(\frac{3.66}{1.38}\right) = 69^\circ \text{EofN} \Rightarrow 021^\circ$.

3.9 km at bearing 021°

Example 4 Using the sine rule

In $\triangle ABC$, $a = 7\text{cm}$, $b = 9\text{cm}$, $A = 38^\circ$. Find angle B .

$$\frac{\sin B}{b} = \frac{\sin 38^\circ}{7} \Rightarrow \sin B = \frac{9 \sin 38^\circ}{7} = 0.784 \Rightarrow B = 51.7^\circ.$$

$B \approx 51.7^\circ$

Example 5 Cosine rule for obtuse angle

Sides of $\triangle PQR$: $PQ = 6\text{cm}$, $PR = 5\text{cm}$, $QR = 9\text{cm}$. Find $\angle P$.

$$9^2 = 5^2 + 6^2 - 2(5)(6) \cos P \Rightarrow \cos P = -0.222. \angle P = 102.8^\circ.$$

103° (nearest degree)

Example 6 Arc length and sector area

Circle radius $r = 4.2\text{cm}$, central angle $\theta = 1.8\text{rad}$. $\ell = r\theta = 7.56\text{cm}$; $A = \frac{1}{2}r^2\theta = 15.9\text{cm}^2$.

$\ell = 7.56 \text{ cm}, A = 15.9 \text{ cm}^2$

Example 7 Simplifying with identities

Simplify $\sin(\pi - \theta) \cos \theta + \sin \theta \cos(\pi - \theta)$.

$$\sin(\pi - \theta) = \sin \theta, \cos(\pi - \theta) = -\cos \theta. \text{ Expression } \Rightarrow \sin \theta \cos \theta - \sin \theta \cos \theta = 0.$$

0

Example 8 Proving the Pythagorean identity

Show $\sin^2 \theta + \cos^2 \theta = 1$ using unit circle. On the circle $x^2 + y^2 = 1$, take $(x, y) = (\cos \theta, \sin \theta)$. Then $\cos^2 \theta + \sin^2 \theta = 1$ as required.

Identity proven

Example 9 Solving a trigonometric equation

Solve $2 \cos^2 x = 1$ for $0^\circ \leq x < 360^\circ$.

$$\cos^2 x = \frac{1}{2} \Rightarrow \cos x = \pm \frac{\sqrt{2}}{2}. \quad x = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$$

$$45^\circ, 135^\circ, 225^\circ, 315^\circ$$

Example 10 Quadratic trig equation

Solve $3\sin^2 t - 4\sin t + 1 = 0$ for $0 \leq t < 2\pi$.

Factor $(3\sin t - 1)(\sin t - 1) = 0 \Rightarrow \sin t = \frac{1}{3}$ or 1. Solutions: $t = \sin^{-1}\frac{1}{3}$, $\pi - \sin^{-1}\frac{1}{3}$, $\frac{\pi}{2}$.

Numerically 0.339, 2.803, 1.571rad.

$$t \approx 0.34, 1.57, 2.80 \text{ rad}$$