Differential Calculus – Worked Examples

Key Facts / Formulas

- Power rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for real n.
- Exponentials and logs: $\dfrac{d}{dx}(e^x) = e^x$, $\dfrac{d}{dx}(a^x) = a^x \ln a$, $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ for $x > 0$.
- Trigonometric: $\dfrac{d}{dx}(\sin x) = \cos x$, $\dfrac{d}{dx}(\cos x) = -\sin x$, $\dfrac{d}{dx}(\tan x) = \sec^2 x$.
- Product rule: $(uv)' = u'v + uv'$. Quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.
- Chain rule: $\dfrac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.
- Implicit: treat y as a function of x and use $\dfrac{d}{dx}(y) = \dfrac{dy}{dx}$.
- Parametric: if $x = x(t)$, $y = y(t)$, then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt}$.
- Second derivative test: if $f'(a) = 0$, $f''(a) > 0$ local min, $f''(a) < 0$ local max. Point of inflection when concavity changes (sign change in f'').

Example 1 Differentiate a polynomial and evaluate a gradient

Find $f'(x)$ and the gradient at $x = 2$ for $f(x) = 4x^5 - 3x^3 + 2x - 7$.

$f'(x) = 20x^4 - 9x^2 + 2$. Then $f'(2) = 20(16) - 9(4) + 2 = 320 - 36 + 2 = 286$.

$$\boxed{f'(x) = 20x^4 - 9x^2 + 2, \quad f'(2) = 286}$$

Example 2 Product rule with trig and exponential

Differentiate $y = (2x^2 - 5)\, e^{3x}$.

$u = 2x^2 - 5 \Rightarrow u' = 4x$, $v = e^{3x} \Rightarrow v' = 3e^{3x}$.
$y' = u'v + uv' = 4x\, e^{3x} + (2x^2 - 5)(3e^{3x}) = e^{3x}(4x + 6x^2 - 15)$.

$$\boxed{y' = e^{3x}\left(6x^2 + 4x - 15\right)}$$

Example 3 Quotient rule rational function

Differentiate $y = \dfrac{3x^2 - 4x + 1}{x^3}$. Simplify.

Use quotient or rewrite: $y = (3x^2 - 4x + 1)x^{-3} = 3x^{-1} - 4x^{-2} + x^{-3}$.
$y' = -3x^{-2} + 8x^{-3} - 3x^{-4} = \dfrac{-3x^2 + 8x - 3}{x^4}$.

$$\boxed{y' = \frac{-3x^2 + 8x - 3}{x^4}}$$

Example 4 Chain rule: power of a trig function
Differentiate $y = (\sin 4x)^5$.
$y = [\sin(4x)]^5$, so $y' = 5[\sin(4x)]^4 \cdot \cos(4x) \cdot 4 = 20 \sin^4(4x) \cos(4x)$.

$$\boxed{y' = 20 \sin^4(4x) \cos(4x)}$$

Example 5 Chain rule: logarithm of a quadratic
Differentiate $y = \ln(5 - 2x - 3x^2)$.
$y' = \dfrac{1}{5 - 2x - 3x^2} \cdot (-2 - 6x) = \dfrac{-2(1 + 3x)}{5 - 2x - 3x^2}$.

$$\boxed{y' = \frac{-2(1 + 3x)}{5 - 2x - 3x^2}}$$

Example 6 Implicit differentiation and slope at a point
For the curve $x^2 + xy + y^2 = 7$, find $\dfrac{dy}{dx}$ and the slope at $(2, 1)$.
Differentiate: $2x + (x\,y' + y) + 2yy' = 0 \Rightarrow (x + 2y)y' = -(2x + y)$.
Thus $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$. At $(2, 1)$: $y' = -\dfrac{4 + 1}{2 + 2} = -\dfrac{5}{4}$.

$$\boxed{\frac{dy}{dx} = -\frac{2x + y}{x + 2y}, \quad y'(2, 1) = -\tfrac{5}{4}}$$

Example 7 Second derivative and point of inflection test
Given $f(x) = x^4 - 4x^2 + 3$, find $f'(x)$, $f''(x)$, and determine whether $x = 0$ is a local extremum or a point of inflection.
$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$, $f''(x) = 12x^2 - 8$. Then $f'(0) = 0$, $f''(0) = -8 < 0$ so $x = 0$ is a local maximum (no inflection, concavity does not change sign at 0).

$$\boxed{f'(x) = 4x(x^2 - 2),\ f''(x) = 12x^2 - 8,\ x = 0 \text{ is a local maximum}}$$

Example 8 Parametric differentiation (first and second derivatives)
For $x = t^2 + 1$, $y = t^3 - 3t$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = 1$.
$\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2 - 3$. Thus $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t} = \dfrac{3(t^2 - 1)}{2t}$. At $t = 1$, $dy/dx = 0$.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(6t)(2t)-(3t^2-3)(2)}{(2t)^2} = \frac{12t^2-2(3t^2-3)}{4t^2} = \frac{12t^2-6t^2+6}{4t^2} = \frac{6t^2+6}{4t^2} = \frac{3(t^2+1)}{2t^2}.$$

Then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{dt} = \dfrac{\frac{3(t^2+1)}{2t^2}}{2t} = \dfrac{3(t^2+1)}{4t^3}$. At $t=1$: $d^2y/dx^2 = \dfrac{3(2)}{4} = \dfrac{3}{2}$.

$$\boxed{\left.\frac{dy}{dx}\right|_{t=1} = 0, \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \tfrac{3}{2}}$$

Example 9 Logarithmic differentiation

Differentiate $y = \dfrac{(x^2+1)^5\sqrt{3x-1}}{e^{2x}}$.

Take logs: $\ln y = 5\ln(x^2+1) + \frac{1}{2}\ln(3x-1) - 2x$. Differentiate:

$$\frac{y'}{y} = \frac{5\cdot 2x}{x^2+1} + \frac{1}{2}\cdot\frac{3}{3x-1} - 2 = \frac{10x}{x^2+1} + \frac{3}{2(3x-1)} - 2.$$

Hence

$$y' = y\left(\frac{10x}{x^2+1} + \frac{3}{2(3x-1)} - 2\right) = \frac{(x^2+1)^5\sqrt{3x-1}}{e^{2x}}\left(\frac{10x}{x^2+1} + \frac{3}{2(3x-1)} - 2\right).$$

$$\boxed{y' = \frac{(x^2+1)^5\sqrt{3x-1}}{e^{2x}}\left(\frac{10x}{x^2+1} + \frac{3}{2(3x-1)} - 2\right)}$$

Example 10 Tangent and normal for a log curve

Find the equations of the tangent and normal to $y = \ln(x^2+1)$ at $x = 2$.

$y' = \dfrac{1}{x^2+1}\cdot 2x = \dfrac{2x}{x^2+1}$, so at $x=2$, slope $m = \dfrac{4}{5}$. Point $(2, \ln 5)$.

Tangent: $y - \ln 5 = \dfrac{4}{5}(x-2)$. Normal slope $= -\dfrac{5}{4}$: $y - \ln 5 = -\dfrac{5}{4}(x-2)$.

$$\boxed{\text{Tangent } y - \ln 5 = \tfrac{4}{5}(x-2), \quad \text{Normal } y - \ln 5 = -\tfrac{5}{4}(x-2)}$$